Three Problems in Robotics

J.M. Selig

South Bank University
London SE1 0AA, U.K.

Abstract

Three rather different problems in robotics are studied using the same technique from screw theory. The first problem concerns systems of springs. We differentiate the potential function in the direction of an arbitrary screw to find the equilibrium position. The second problem is almost identical in terms of the computations, we seek the least squares solution to the problem of finding the rigid motion undergone by a body given only data about points on the body. Finally we look at the Jacobian of a Stewart platform. Six point on the moving platform are connected by prismatic joints to six fixed points on the base. The Jacobian relates the joint rates of the legs to the linear and angular velocities of the platform. Again this can be done by differentiating with respect to a screw.

Although all three problems are quite well known and have be solved by other means, the screw theory method is significantly simpler. For example, when minimising a potential or residual function we do not need to include constraints. These advantages mean that we can go further and study second order properties, the Hessian of the spring system, whether or not the least squares solution minimises or maximises the residual.

1 Introduction

When Ball wrote his treatise at the end of the 19th century Sophius Lie was just beginning to think about what he called ‘continuous groups’. If Ball knew of Lie’s work it may not have been obvious that it had any connection to his own since Lie was interested in symmetries of differential equations. It was Klien who later introduced the idea that these ‘Lie groups’ could be thought of as geometrical symmetries. It was after both Ball and Lie had died that ‘Lie theory’ began to finds it place as central to modern geometry. In particular, the work of Killing and Cartan on Lie algebras were very influential.

With hindsight we can see that Ball’s finite screws were simply elements of a Lie group: the group of proper Euclidean transformations in \( \mathbb{R}^3 \).
The infinitesimal screws or motors were elements of the Lie algebra of this group. Many other elements of Lie theory were also present in Ball’s screw theory. But perhaps their significance was not fully appreciated. For example, the Lie product or Lie bracket is simply the cross product of screws. For Ball this was just a geometrical operation, the analogue of the vector product of 3-dimensional vectors.

Some authors refer to Screw theory and Lie group methods as if they were different approaches. This author’s view is that there is no distinction between them, screw theory is simply the specialisation of Lie theory to the group of rigid body transformations. The name screw theory is useful in three ways, it is useful shorthand, it is descriptive and lastly it reminds us that it was Ball who worked out almost all of the theory before Lie groups were invented!

In this work the fact that a Lie group is a differential manifold is used. To minimise a smooth function on such a space we do not need the machinery of Lagrange multipliers. We can work on the manifold directly, we do not need to think of the group as embedded in Cartesian space, as would be implied by the use of Lagrange multipliers.

To find equations for a function’s stationary points we differentiate along tangent vector fields. The most convenient vector fields to use are the left-invariant fields on the group. These are just elements of the Lie algebra of the group, the screws. Hence this technique could be thought of as, “differentiating along a screw”.

2 Springs

Consider a rigid body supported by a system of springs, see figure 1. Further assume that the springs have natural length 0, obey Hook’s law and can both push and pull. The spring constants $\lambda_i$ of the springs can be different. Let $\tilde{a}_i = (a_i^T, 1)$ be the points where the springs are attached to the ground or frame, and $\tilde{b}_i = (b_i^T, 1)$ the corresponding attachment points on the rigid body when the body is in some standard ‘home’ config-

![Figure 1: A Rigid Body Suspended by a System of Springs](image)
uration. If the body undergoes a rigid motion the attachment points will move to,

\[
\begin{pmatrix}
\mathbf{b}_i' \\
1
\end{pmatrix} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix}
\mathbf{b}_i \\
1
\end{pmatrix}
\]

which we will abbreviate to \( \mathbf{b}_i' = M \mathbf{b}_i \). The first question we can ask about this situation is: Is there an equilibrium configuration for the rigid body?

The problem is to minimise the potential energy of the spring system, this is given by the following function,

\[
\Phi = \frac{1}{2} \sum_i \lambda_i (\mathbf{a}_i - M \mathbf{b}_i)^T (\mathbf{a}_i - M \mathbf{b}_i)
\]

Notice that this function is defined on the group \( SE(3) \), that is \( \Phi: SE(3) \rightarrow \mathbb{R} \), as \( M \) varies over the group we have different values of the potential.

If this was just a function on \( \mathbb{R}^n \) we would find the partial derivatives and set them to zero to find the stationary points. The standard method of tackling this problem would be to minimise the matrix elements using Lagrange multipliers to take account of the constraint that the matrix must be a group element.

However, we can imitate the simpler method for unconstrained functions using some manifold theory. To find the stationary points of a function defined on a manifold we simply differentiate the function along a vector fields on the manifold. As usual we set the results to zero and then solve the resulting equations to find the stationary point. For this to work we do need a set of vector field which span the space of all vector field on the manifold.

As the manifold under consideration is the underlying manifold of a Lie group, such a complete set of vector fields is always available. We can use the elements of the Lie algebra thought of a left invariant vector fields.

To differentiate along a vector field we move a little way along a path tangent to the vector field and compare the function values at the two points, then we take the limit of the difference between values of the function at these neighbouring points as the path get shorter and shorter.

Let:

\[
S = \begin{pmatrix} \Omega & \mathbf{v} \\ 0 & 0 \end{pmatrix}
\]

be a Lie algebra element or screw given in the \( 4 \times 4 \) representation. \( \Omega \) is a \( 3 \times 3 \) anti-symmetric matrix corresponding to a vector \( \omega \), \( \Omega \mathbf{x} = \omega \times \mathbf{x} \).

If \( M \) is a group element written in the \( 4 \times 4 \) representation, then the action of \( S \) on \( M \), is given by the left translation,

\[
M(t) = e^{tS} M
\]

This takes \( M \) along a path tangent to the vector field defined by \( S \). Taking the derivative along the path and then setting \( t = 0 \) gives,

\[
\partial_S M = SM
\]

Hence the derivative of the potential is given by,

\[
\partial_S \Phi = - \sum_i \lambda_i (\mathbf{a}_i - M \mathbf{b}_i)^T SM \mathbf{b}_i
\]
Now for equilibrium this must vanish for arbitrary $S$. Hence we must separate out $S$, to do this look at the term,

$$SM\tilde{b}_i = \begin{pmatrix} \omega \times (Rb_i + t) + v \\ 0 \end{pmatrix} = \begin{pmatrix} -RB_i R^T - T \\ 0 \end{pmatrix} \begin{pmatrix} \omega \\ v \end{pmatrix}$$

where $I_3$ is the $3 \times 3$ identity matrix and $B_i$ is the anti-symmetric matrix corresponding to $b_i$.

If we substitute this into the equilibrium equation and use the fact that $S$ and thus $\omega$ and $v$ are arbitrary, we get,

$$\sum_i \lambda_i (\tilde{a}_i - M\tilde{b}_i)^T \begin{pmatrix} -RB_i R^T - T \\ 0 \end{pmatrix} = 0$$

After a little manipulation, this matrix equation produces 2 vector equations,

$$\sum_i \lambda_i a_i \times (a_i - Rb_i - t) = 0 \quad (1)$$

and

$$\sum_i \lambda_i (a_i - Rb_i - t) = 0 \quad (2)$$

If the weights $\lambda_i$ are all equal, then equation 2 tells us that the optimal transformation maps the centroids of the $a$ points to the $b$ points. Another way of putting this is that at an equilibrium configuration the centroids of the two sets of points must coincide. To proceed, we choose the origin of coordinates so that the centroid of the $b$ points lies at the origin, $\sum_i \lambda_i b_i = 0$. The translation vector is now given by equation (2) as,

$$t = \frac{\sum_i \lambda_i a_i}{\sum_i \lambda_i}$$

In the above form equation (1) is not very easy to deal with, a more tractable form is the $3 \times 3$ representation. A small computation confirms that the anti-symmetric matrix corresponding to a vector product $p \times q$, is given by $qp^T - pq^T$. Hence in this form the equation becomes,

$$\sum_i \lambda_i (Rb_i a_i^T - a_i b_i^T R^T) = 0$$

Now, writing, $P = \sum_i \lambda_i a_i b_i^T$ and using the result that $t = \sum_i \lambda_i a_i / \sum_i \lambda_i$, this equation becomes;

$$RP^T = PR^T \quad (3)$$

This shows that the matrix $PR^T$ is symmetric. So let us write $PR^T = Q$ where $Q$ is symmetric, then we have that,

$$P = QR$$

This decomposes the matrix $P$ as the product of a symmetric matrix with a proper orthogonal one. This is, essentially the polar decomposition of the of the matrix. Notice that the polar decomposition $P = RQ'$ also satisfies the equation, the rotation matrix $R$ here is the same as above but the symmetric matrix $Q' = R^T Q R$ is simply congruent to the original symmetric matrix. So as far as the solution for $R$ is concerned there is no difference between these solutions. In fact the polar decomposition of a matrix is
into an orthogonal matrix and a non-negative symmetric matrix. But here we want a proper orthogonal matrix and a symmetric one. If the orthogonal matrix from the polar decomposition of \( P \) is a reflection then we can simply multiply by \(-1\) to get a rotation. More details on the polar decomposition of a matrix can be found in [6], for example.

The polar decomposition give one solution, but this solution is not unique. Let \( P = QR_p \) be the polar decomposition of \( P \), now substitute this into the equation (3),

\[
RR_p^TQP^T = QR_pR^T
\]

If we write \( R_i = RR_p^T \) then the equation becomes,

\[
R_iQR_i = Q
\]

Suppose that \( v \) lies in the direction of the rotation axis of \( R_i \), so that \( R_i v = v \), postmultiplying the above equation by \( v \) gives,

\[
R_iQv = Qv
\]

Hence \( Qv \) lies along the axis of \( R_i \) and we can write,

\[
Qv = \mu v
\]

for some constant \( \mu \). Any solution for the rotation \( R_i \) must have its axis of rotation aligned with an eigenvector of \( Q \).

The possible angles of rotation can be found by considering the action on the eigenvectors of \( Q \), using the fact that the eigenvectors of a symmetric matrix are mutually orthogonal. If the matrix \( P \) is non-singular then it is well known that the polar decomposition is unique. If the eigenvalues of \( Q \) are all different and also have different magnitudes then the only possible angles are 0 and \( \pi \). This gives four solutions in all, \( R_i = I_3 \) is the solution we found above, that is \( R \) is simply the rotation from the polar decomposition. The three other solutions for \( R_i \) are rotations of \( \pi \) radians about the three eigenvectors of \( Q \). So in all we have four solutions \( R = R_pR_i \), where \( R_p \) is the rotation from the polar decomposition of \( P \) and \( R_i \) are as above, rotations of \( \pi \) about the \( i \)th eigenvector of \( Q \), the fourth solution is given by \( R_0 = I_3 \). Notice that there four rotations form a discrete subgroup of the group of rotation, this subgroup is the symmetry group of the octahedron.

If any of the above conditions is broken, \( P \) is singular, two of the eigenvalues of \( Q \) are equal, or a pair of eigenvalues sum to zero, then we get more solutions. For example, if a pair of eigenvalues of \( Q \) sum to zero then any rotation about the remaining eigenvector will satisfy the equation for \( R_i \).

The fact that, in the general case, we have found four stationary points of the potential energy function is not surprising. Morse theory studies the relationship between manifolds and the critical points of functions defined on them, see [9]. The critical points, or stationary points here, correspond to cells in a cellular decomposition of the manifold. The manifold in question here is the underlying manifold of the rotation group \( SO(3) \), this is known to be 3-dimensional projective space, \( \mathbb{RP}^3 \). The minimal cellular decomposition of \( \mathbb{RP}^3 \) has for cells, with dimensions 0, 1, 2 and 3 see [3, p. 105]. Hence, we expect a minimum of 4 critical point for function on \( SO(3) \). Moreover, index of the critical point, the number of negative eigenvalues of its Hessian
matrix, gives the dimension of the corresponding cell. Thus, without any further computations, we can say that among the four critical points will be a local maxima a local minima and two types of saddle points. We will return to the problem of finding which solution is the minima later.

3 Rigid motion from point data

Suppose we have a vision system or range-finding system, which can measure the location of points in 3-dimensions. Imagine that we have a rigid body and know the position of a number of point on the body. The body is subjected to an unknown rigid motion and the positions of the points are measured. These measurements will contain errors and the question we want to address here is: How can we estimate the rigid motion undergone by the body?

Let us represent the positions of the known points as $\mathbf{b}_i$ and corresponding measured points are $\mathbf{a}_i$. Write the unknown rigid transformation as $M$, then the function,

$$\Phi = \sum_i (\mathbf{a}_i - M\mathbf{b}_i)^T (\mathbf{a}_i - M\mathbf{b}_i)$$

represents the sum of squares of the differences between the measured points and their ideal (noise-free) positions. Choosing $M$ to minimise this function gives a ‘least-squares’ estimate for the rigid transformation. This function is almost identical to the potential energy function studied in the previous section, the only differences are an overall factor of one half and that all the $\lambda_i$s have been set to 1.

The history of this problem is very interesting. The problem of finding the rotation is clearly the interesting part and was first solved by MacKenzie in 1957 [8]. He came upon this problem in the context of crystallography. In 1966 Wahba found the same problem while studying the orientation of artificial satellites, [16]. In 1976 Moran re-solved the problem using quaternions, [10]. The motivation here came from geology, in the particular the movement of tectonic plates. In the context of manufacturing, Nádas found and re-solved the problem in 1978, [11]. Here the application was to the manufacture of ceramic substrates for silicon chips. In the robot vision community the problem is usually credited to Horn [5], for example see [7, Chap. 5].

The solution given above is, perhaps, a little simpler than the standard arguments which involve a constrained minimisation, the constraints being used to express the fact that the matrices must lie in the group.

The standard solutions have not always been in terms of the polar decomposition. In fact (at least), two other descriptions of the solution are possible.

To compute the polar decomposition of a matrix, texts on numerical analysis advise us to start with the singular value decomposition of the matrix, see for example [12]. Hence, it is no real surprise that we can obtain the solution to our problems directly from a singular value decomposition.

Recall, from the section above that we are required to solve $RP^T = PR^T$, for the rotation matrix $R$, where $P = \sum_i \lambda_i a_i b_i^T$. Further, recall that,

$$P = QR_p$$

where $Q$ was symmetric. Hence, we can diagonalise $Q$ as $Q = UDU^T$ with $U$ orthog-
onal and $D$ diagonal. Now we can write,

$$P = UDV^T R_p = UDV^T$$

where $V^T = U^T R$ is still orthogonal. This is simply the singular value decomposition of $P$. To put this another way, suppose the singular value decomposition of $P$ is $P = UDV^T$ then we have the four solutions $R = UV^T R_i$.

We can derive another form for the solution as follows, we begin with the polar decomposition of $P$,

$$P = QR_p$$

where $Q$ is a symmetric matrix. Postmultiplying this equation by its transpose gives, $P P^T = Q^2$. Finally substituting for $Q$ yields,

$$R = (P P^T)^{-1/2} P R_i$$

There are several different square roots of the matrix $(P P^T)$ that we could take here, the choice is limited by the requirement that the determinant of $R$ must be 1. This means we must take the unique positive square root, see [6, p. 405].

Finally here we look at the determinant of the matrix $P$. A classical result tells us that the polar decomposition of a matrix $P$ is unique if $P$ is non-singular, see [6, p. 413] for example. The classic polar decomposition, decomposes the matrix $P$ into an orthogonal matrix and positive-semidefinite symmetric matrix. For our purposes we need a proper-orthogonal matrix and a symmetric matrix (not necessarily positive-semidefinite), this does not effect the uniqueness of the solution.

Thus we are led to investigate the determinant of the matrix $P$ defined above. To simplify the discussion we will assume that the spring stiffnesses $\lambda_i$ have all been set to 1.

When there are less than three springs or pairs of points the determinant is always singular. For three point-pairs a straightforward computation reveals,

$$\det(P) = \det \left( \sum_{i=1}^{3} a_i b_i^T \right) = (a_1 a_2 a_3)(b_1 b_2 b_3)$$

here the scalar triple product has been written as, $a \cdot (b \times c) = (abc)$.

Generalising this to $n$ point-pairs we have that,

$$\det(P) = \sum_{1 \leq i < j < k \leq n} (a_i a_j a_k)(b_j b_k b_k)$$

Certainly this is singular if all the $a$ points or all the $b$ points lie on a plane through the origin.

## 4 Jacobian matrix for Stewart platforms

Consider a general Stewart platform. This manipulator has six legs connected in parallel. Each leg consists of an hydraulic actuator between a pair of passive spherical joints. The six leg connect the base or ground to a movable platform. By adjusting the lengths of the six legs using the hydraulic actuators the platform can be manoeuvred with six degrees-of-freedom. See figure 2
Figure 2: A General Stewart Platform

For parallel manipulators it is the inverse kinematics that is straightforward, while the forward kinematics are hard. Suppose that we are given the position and orientation of the platform the leg-lengths are simple to find. Let us write \( \mathbf{a}_i \) for the position of the centre of the spherical joint on the ground belonging to the \( i \)-th leg. In the home configuration the corresponding position of the joint centre on the platform will be, \( \mathbf{b}_i \). So now the length of the \( i \)-th leg, or rather its square, can be written,

\[
l_i^2 = (\mathbf{\tilde{a}}_i - M\mathbf{\tilde{b}}_i)^T (\mathbf{\tilde{a}}_i - M\mathbf{\tilde{b}}_i) \quad i = 1, \ldots, 6
\]

As usual \( M \) is a rigid transformation, this time the motion that takes the platform from home to the current position. Notice that the leg-lengths can be thought of as functions on the group, however it is more usual to think of these as components of a mapping from the group to the space of leg-lengths, \( SE(3) \rightarrow \mathbb{R}^6 \). A point in \( \mathbb{R}^6 \) is given in coordinates as \( (l_1, l_2, \ldots, l_6) \). It is the jacobian of this mapping that we seek. To do this we take the derivative of the leg-lengths,

\[
\frac{dl_i^2}{dt} \bigg|_{t=0} = 2l_i \dot{l}_i = -2(\mathbf{\tilde{a}}_i - \mathbf{\tilde{b}}_i)^T S\mathbf{\tilde{b}}_i
\]

Rearranging this gives,

\[
\dot{l}_i = \frac{1}{l_i} (\mathbf{b}_i - \mathbf{\tilde{a}}_i)^T S\mathbf{\tilde{b}}_i = \frac{1}{l_i} (\mathbf{a}_i \times \mathbf{b}_i)^T, (\mathbf{b}_i - \mathbf{a}_i)^T \begin{pmatrix} \omega \\ v \end{pmatrix}
\]

This gives the joint rate of each leg as a linear function of the velocity screw of the platform. The jacobian \( J \) is the matrix satisfying the formula,

\[
\begin{pmatrix} \dot{l}_1 \\ \vdots \\ \dot{l}_6 \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1 \times \mathbf{b}_1 \\ \mathbf{a}_2 \times \mathbf{b}_2 \\ \vdots \\ \mathbf{a}_6 \times \mathbf{b}_6 \end{pmatrix} \begin{pmatrix} \omega \\ v \end{pmatrix}
\]
So we see that the rows of this Jacobian matrix are simply,
\[
\frac{1}{l_i}((a_i \times b_i)^T, (b_i - a_i)^T), \quad i = 1, \ldots, 6
\]
This is the wrench given by a unit force directed along the \(i\)-th leg.

Consider a system of springs as in section 2. Suppose there are just six springs. Now it can be shown that the Jacobian associated with the equilibrium position is singular. To see this consider equations (1) and (2), which define the equilibrium position, if we arrange things so that the equilibrium position is the reference position and hence \(R = I\) and \(t = 0\) then the equations become,
\[
\sum \lambda_i a_i \times b_i = 0
\]
and
\[
\sum \lambda_i (a_i - b_i) = 0
\]
In term of the Jacobian for a corresponding Stewart platform, that is, one whose leg-lengths correspond to the lengths of the springs, we see that the rows of the Jacobian are linearly dependant and hence the matrix is singular.

The forward kinematics problem for a Stewart platform is to determine the position and orientation of the platform given the leg-lengths. It is well known that, for a given set of leg-lengths there are a finite number of different solutions, in general 40. Different solutions are referred to as different poses or postures of the platform. Replacing the legs with springs it is clear that the potential function will have the same value in each of these poses or postures since the function only depends on the lengths of the springs. However, we can also see that none of these positions will be minima of the potential function since we know the stable equilibrium position is unique.

5 The Stiffness Matrix

The problems presented in the last three sections are well known and have been solved by many different methods. The advantage of the screw theory methods studied is that we can go further and look at higher derivatives.

First we note that it is possible to find the wrench due to the springs. In general a wrench is a 6-dimensional vector of forces and torques,
\[
W = \begin{pmatrix} \tau \\ F \end{pmatrix}
\]
where \(\tau\) is a moment about the origin and \(F\) is a force. Notice that wrenches are not Lie algebra elements but elements of the vector space dual to the Lie algebra. Usually the force due to a potential is given by its gradient. The same is true here, in terms of the exterior derivative \(d\) we have, \(W = -d\Phi\). Pairing the wrench with an arbitrary screw \(S\) gives,
\[
W(S) = -d\Phi(S) = -\partial_S \Phi
\]
see [13, § 4.20] for example. Hence we have already done the calculations in section 2 above, the wrench is given by,
\[
W = \left( \sum \lambda_i a_i \times (a_i - Rb_i - t) \right) \left( \sum \lambda_i (a_i - Rb_i - t) \right)
\]
this could, of course, also have been deduced from elementary mechanics.

For the spring systems of section 2 an important object is the stiffness matrix of the system. In this section we show how to compute this by taking the second derivatives of the potential function.

An infinitesimal displacement of the body is represented by a screw. The wrench produced by a displacement \( s \) is given by \( W = Ks \), where \( K \) is the stiffness matrix.

The stiffness matrix is the hessian of the potential function, that is its matrix of second order partial derivatives, see [1, Ch 5]. This is only valid at an equilibrium position.

There have been attempts in the Robotics literature to extend these ideas to non-equilibrium configurations, see for example Griffis and Duffy [4] and Žefran and Kumar [18]. In this work, however, the classical definition of the stiffness matrix will be used.

Now let us write the result above for the wrench as

\[
W = \sum \lambda_i \begin{pmatrix} A_i & 0 \\ I_3 & 0 \end{pmatrix} (\bar{a}_i - M\bar{b}_i)
\]

Differentiating this along an arbitrary screw gives,

\[
\frac{\partial S}{\partial s} W = \sum \lambda_i \begin{pmatrix} A_i & 0 \\ I_3 & 0 \end{pmatrix} \begin{pmatrix} RB_i R^T + T & I_3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega \\ v \end{pmatrix}
\]

using the result of section 2. Hence, we have that the stiffness matrix is,

\[
K = \begin{pmatrix} \sum \lambda_i (A_i RB_i R^T + A_i T) & \sum \lambda_i A_i \\ \sum \lambda_i (RB_i R^T + T) & \sum \lambda_i I_3 \end{pmatrix}
\]

This time we choose the origin to be at the point \( \sum i \lambda_i a_i \) and then use the equilibrium condition 2 to simplify the stiffness matrix to,

\[
K = \begin{pmatrix} \sum \lambda_i A_i RB_i R^T & 0 \\ 0 & \sum \lambda_i I_3 \end{pmatrix}
\]

This is a particularly neat result but it is a little surprising at first sight. The term \( \sum \lambda_i I_3 \) in the bottom right corner means that the system has the same stiffness in any direction, irrespective of the stiffness of the individual springs and their arrangement. This is due to the assumption that the springs have zero natural length.

Now we can return to the problem of finding the index of the critical points we have found. For brevity we will just look at which of the critical points is a minimum of the potential energy, that is a stable equilibrium. That is: which of the solutions for \( R \) gives a stiffness matrix \( K \) with all positive eigenvalues? Notice that three of the eigenvalues of \( K \) are simply \( \sum \lambda_i \), and this is positive if the spring constants \( \lambda_i \) are all positive. So we just have to look at the eigenvalues of the top-lefthand bock of \( K \). After a little manipulation this can be written in term of the matrix \( P \) or its polar decomposition,

\[
\sum \lambda_i A_i RB_i R^T = RP^T + Tr(RP^T)I_3 = R_i Q + Tr(R_i Q)I_3
\]

Here \( Tr(\cdot) \) represents the trace of the matrix. Now this matrix has the same eigenvectors as \( R_i Q \) and hence as \( Q \), remember \( R_i \) was either the identity or a rotation of \( \pi \) about
an eigenvector of $Q$. So let us assume that the eigenvalues of $Q$ are $\mu_1, \mu_2$ and $\mu_3$ with corresponding eigenvectors $e_1$, $e_2$ and $e_3$. Assuming that $R_0$ is a rotation about eigenvector $e_i$, we can find the eigenvalues of the matrices by considering the action on the eigenvectors $e_1$, $e_2$ and $e_3$. The eigenvalues for $R_0Q + \text{Tr}(R_0Q)I_3$ are,

$$(2\mu_1 + \mu_2 + \mu_3), \quad (\mu_1 + 2\mu_2 + \mu_3), \quad \text{and} \quad (\mu_1 + \mu_2 + 2\mu_3)$$

We only need to look at one other matrix since the other are just cyclic permutations; the eigenvalues of $R_1Q + \text{Tr}(R_1Q)I_3$ are,

$$(2\mu_1 - \mu_2 - \mu_3), \quad (\mu_1 - 2\mu_2 - \mu_3), \quad \text{and} \quad (\mu_1 - \mu_2 - 2\mu_3)$$

Now there are just two cases to consider, if $\det(P) > 0$ then we can assume that $Q$ is positive-definite and thus so are all its eigenvalues. In this case it is easy to see that the critical point represented by $R_0$ will be the minimum. This is the solution $R = R_p$.

In the other case $\det(P) < 0$, the classical polar decomposition gives us a positive-definite symmetric matrix and a reflection. When we multiply these by $-1$ we get a rotation and a negative-definite symmetric matrix. That is $Q$ has all negative eigenvalues. If we assume that these eigenvalues have the ordering $0 > \mu_1 \geq \mu_2 \geq \mu_3$, that is $\mu_1$ is the eigenvalue of smallest magnitude, then the matrix, $R_1Q + \text{Tr}(R_1Q)I_3$ will have all positive eigenvalues and hence $R_1$ corresponds to the minimum of the potential. So in general, if $\det(P) < 0$ the minimum solution is given by $R = R_p R_i$, where $R_i$ is a rotation of $\pi$ about the eigenvector of $Q$ with eigenvalue of smallest magnitude.

## 6 Conclusions

The concept unifying the three problems we have studied in this work is the idea of functions defined on the group of rigid body transformations. We have been able to find the stationary points of the functions and classify these critical points. This involves a simple technique of differentiating along a screw.

The results agree with those of Kanatani [7, Chap. 5], his methods were, perhaps, a little more elegant than the above. But the methods used here are more general, we have found all the critical points of the potential function not just the minimum and with a little more effort we could find the index of each of them.

These ideas are central to the subject of Morse theory, a field of study which relates the critical points of functions defined on a manifold to the topology of the manifold itself. In this case the topology of the manifold, the underlying manifold of the group $SO(3)$, is well known, and we can use this to say something about the critical point of our potential function.

The problem of the springs is a slightly artificial one in that we have used springs will zero natural length. This simplifies the computations. It is possible to repeat most of the above analysis using spring with a finite natural length, see [14]. The number of critical points now becomes a very hard problem but will still be constrained by the topology of the group.

The problem of estimating a rigid motion from point data is reasonably realistic. The utility of the estimate will depend on the distribution of error for the measured points. There are some results on this in the literature, see [17]. However, I believe there is much scope for further research in this direction.
We have only had time here to take a brief look at the implications for the Stewart platform. Certainly using the technique of differentiating along a screw we could work out the acceleration properties of the Stewart platform. However, the usual difficulties on the geometric definition of a second derivative arise here. In particular circumstances it is clear what should be done, for example it is not too difficult to find the dynamics of a Stewart platform, see [15]. Once again, I believe that the implications of the ideas presented here have yet to be fully understood.

References


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