ASYMMETRIC CARTESIAN STIFFNESS FOR
THE MODELLING OF COMPLIANT ROBOTIC SYSTEMS

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Abstract

Models for compliant robotic systems often use a symmetric $6 \times 6$ stiffness matrix. However, when subjected to external loads, the stiffness actually becomes asymmetric. For a compliant system modelled using line springs, a new and important theorem is presented that represents the skew-symmetric part in its simplest form: the skew-symmetric part of the stiffness matrix is negative one-half the externally applied load expressed as a spatial cross product operator. Several corollaries follow including the obvious result that the stiffness matrix is symmetric if and only if it is at an unloaded equilibrium.

1 Introduction

Many robotic systems are modelled as rigid bodies connected by elastic couplings. Common examples include robotic fingers grasping an object, walking machines with flexible legs, Stewart platform-type manipulators, remote center-of-compliance devices, and special end-of-arm tooling for constrained motions. These systems can be modelled using a $6 \times 6$ Cartesian stiffness matrix. There are also applications in active control where the position gain matrix is physically analogous to the stiffness matrix.

This paper investigates a fundamental type of elastic coupling between two rigid bodies (see Figure 1). The coupling is modelled as a set of line springs constrained within cylindrical or prismatic joints and connected to each body by ball and socket joints. This allows pure forces to be exerted between the bodies in tension and compression. The individual spring rates are assumed constant but the springs may be preloaded. The system spring rate is modelled by a $6 \times 6$ stiffness matrix which is a function of position. It relates small linear and angular displacements from an initial configuration to small changes in the applied force and torque.

Dinekeberg (1965) first used screw theory to construct the stiffness matrix at an unloaded equilibrium. He characterized the eigensystem which was later extended and generalized by Lipkin and Patterson (1993a, 1993b). Loncaric (1985) used Lie algebra to study the properties of symmetric stiffness matrices. Griffis (1993) derived an asymmetric stiffness matrix for a Stewart platform-type device with six springs. In Pigozki, Griffis, and Duffy (1992) a planar, three spring, elastic coupling was used to investigate mappings of a $3 \times 3$ asymmetric stiffness matrix.

The fundamental contribution of this paper is the reduction of the skew-symmetric part of the stiffness matrix to its simplest and most understandable form. This has practical significance since very compliant systems usually operate away from the unloaded equilibrium configuration resulting in asymmetric stiffness matrices. For example, asymmetry can be caused by the effect of gravity forces alone. Immediate results of the skew-symmetric reduction are: the stiffness matrix
is symmetric if and only if it is at unloaded equilibrium, stiffness matrices composed of line springs in parallel are characterized by 26 independent parameters (not 36), and the stiffness matrices viewed in the fixed and moving frames are transposes.

2 Mathematical Background

The elastic coupling model consists of arbitrarily aligned linear line springs connecting two bodies in three dimensions by spherical joints. Springs are modeled to be confined in prismatic (or cylindrical) joints (Figure 1). One of the bodies is denoted as the fixed body $\mathcal{F}$, and the other as the moving body $\mathcal{F}'$.

A spatial vector is a six-vector composed of two three-vectors and transforms by a group of linear rigid body transformations. Geometric duality is expressed using ray and axis Plücker coordinates, the formal difference being that the order of the three-vectors is interchanged. For example, the wrench $\mathbf{\hat{W}}_{P'} = [ \mathbf{\hat{T}}_{P'} \mathbf{\hat{t}}_{P'} ]^T$ in ray coordinates is composed of a force $\vec{f}$ through point $P'$ and a total moment $\vec{t}_{P'}$ about $P'$. An equipollent system expressed at point $O$ is

$$\mathbf{\hat{W}}_O = \mathbf{\hat{X}}_{O/P'} \mathbf{\hat{W}}_{P'} = \left[ \begin{array}{cc} 1 & 0 \\ OP' \times & 1 \end{array} \right] \left[ \begin{array}{c} \vec{f} \\ \vec{t}_{P'} \end{array} \right]$$

$$= \left[ \begin{array}{c} \vec{f} \\ OP' \times \vec{f} + \vec{t}_{P'} \end{array} \right] = \left[ \begin{array}{c} \vec{f} \\ \vec{t}_O \end{array} \right]$$

A velocity twist in axis coordinates, $\mathbf{\hat{T}}_{P'} = [ \mathbf{\hat{v}}_{P'}, \mathbf{\hat{\omega}} ]^T$ is composed of an angular velocity $\mathbf{\hat{\omega}}$ through point $P'$ and the linear velocity at $P'$. The equivalent rigid body motion expressed at point $O$ is

$$\mathbf{\hat{T}}_O = \mathbf{\hat{X}}_{O/P'} \mathbf{\hat{T}}_{P'} = \left[ \begin{array}{cc} 1 & 0 \\ OP' \times & 1 \end{array} \right] \left[ \begin{array}{c} \vec{v}_{P'} \\ \vec{\omega} \end{array} \right]$$

$$= \left[ \begin{array}{c} \vec{v}_{O} \\ \vec{\omega} \end{array} \right]$$

The linear transformations for ray and axis coordinates are adjoints (i.e., inverse transposes). Rigid body rotations are given by the transformation

$$\mathbf{\hat{X}}_{\mathcal{F}'/\mathcal{F}} = \left[ \begin{array}{cc} R & 0 \\ 0 & R \end{array} \right]$$

where $R$ is a $3 \times 3$ rotation matrix. Since the transformation is self-adjoint, it applies to both ray and axis coordinates.

The derivatives of a three-vector $\vec{z}$ with respect to fixed and moving bodies are related by

$$\frac{d}{dt} \vec{z} = \frac{d'}{dt} \vec{z} + \vec{\omega} \times \vec{z}$$

For spatial vectors this becomes

$$\frac{d}{dt} \vec{z}_{P'} = \frac{d'}{dt} \vec{z}_{P'} + \mathbf{\hat{T}}_{P'} \times \vec{z}_{P'}$$

where $\vec{z}$ is expressed at a point $P'$ in $\mathcal{F}'$. The spatial cross product is detailed in the appendix. Distinct derivations of (5) are in Brand (1947) and Featherstone (1987).

For elastically suspended bodies, it is useful to remove the explicit dependence on time and introduce the (deformation) twist $\mathbf{\hat{d}q}_{P'} = [ \mathbf{\hat{d}T}_{P'} \mathbf{\hat{d}t}_{P'} ]^T$ where $\mathbf{\hat{d}T}_{P'}$ is a small linear displacement of point $P'$ and $\mathbf{\hat{d}t}_{P'}$ is a small rotation of $\mathcal{F}'$ roughly equivalent to $\mathbf{\omega} dt$. Since $\mathbf{\hat{d}t}_{P'}$ is the differential of quasicoordinates, this is indicated by an arrow over the entire differential. Similarly, since $\mathbf{\hat{d}q}_{P'}$ contains the differential of quasi-coordinates, this is indicated by a hat over the entire differential. The twist $\mathbf{\hat{d}q}$ also transforms linearly by $\mathbf{\hat{X}}_{\mathcal{F}'/\mathcal{F}}$ in (2) and (3). The differentiation relations (4) and (5) become

$$d\vec{z} = d'\vec{z} + d\mathbf{\hat{\theta}} \times \vec{z}$$

$$d\vec{z}_{P'} = d'\vec{z}_{P'} + d\mathbf{\hat{T}}_{P'} \times \vec{z}_{P'}$$
3 Stiffness Matrix

Figure 2 shows a wrench $\hat{W}_{P'}$ applied at $P'$ in $\mathcal{T}'$ to an elastically suspended rigid body. It is desired to express all quantities at $O$ in $\mathcal{T}$ and explicitly determine the differential expression

$$d\hat{W}_O = \hat{K}_O \, d\hat{q}_O$$

(8)

where $\hat{K}_O$ is defined as the stiffness matrix. It relates small changes in the applied load to small relative displacements. For simplicity, (8) is derived for a single line spring (see Figure 2) and is then subsequently generalised for an arbitrary number of springs.

First an expression for $d\hat{q}_O$ is derived. Transforming $d\hat{q}_{P'}$ to $O$ gives

$$d\hat{q}_O = \hat{X}_{O/P'} \, d\hat{q}_{P'} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d\hat{r}_{P'}}{d\theta} \\ \frac{d\hat{r}_{P'}}{db} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{O}P' \times \frac{d\hat{r}_{P'}}{d\theta} + d\hat{r}_{P'} \end{bmatrix}$$

(9)

The position vector from $O$ to $P'$ is

$$\hat{r}_{P'} \equiv \hat{O}P' = \hat{O}A + \hat{A}B + \hat{B}P'$$

$$\quad = \hat{O}A + l\hat{s} + \hat{B}P'$$

(10)

where $A$ and $B'$ are the spring connection points, $l$ is the spring length, and $\hat{s}$ is a unit vector in the direction of the spring axis. Differentiating (10) and using (6) to reduce $d\hat{B}P'$ yields

$$d\hat{r}_{P'} = \hat{d}\hat{O}\hat{A} + \hat{d}\hat{s} + l\hat{d}\hat{s} + d\hat{B}P'$$

$$= \hat{d}\hat{r}_{s} + l\hat{d}\hat{s} + \frac{\hat{d}\hat{r}_{P'}}{db} \times \hat{B}P'$$

(11)

where $\hat{d}\hat{O}\hat{A}$, $d\hat{B}P'$, $= 0$. Substituting (11) into (9) gives the kinematic relation,

$$d\hat{q}_O = \begin{bmatrix} \hat{O}B' \times \frac{d\hat{r}_{P'}}{db} + d\hat{r}_{s} + l\hat{d}\hat{s} \end{bmatrix}$$

(12)

Next $\hat{W}_O$ is determined by introducing the spring constant $k$ and free length $l_0$,

$$\begin{bmatrix} \hat{W}_O \end{bmatrix} = \begin{bmatrix} \hat{f} \\ \hat{O}\hat{A} \times \hat{f} \end{bmatrix} = k(l - l_0)\hat{S}_O$$

(13)

where $\hat{S}_O$ is a unit spatial line vector along the spring,

$$\begin{bmatrix} \hat{S}_O \end{bmatrix} = \begin{bmatrix} \hat{s} \\ \hat{O}\hat{A} \times \hat{s} \end{bmatrix} = \begin{bmatrix} \frac{\hat{s}}{\hat{O}B' \times \hat{s}} \end{bmatrix}$$

(14)

Using (12), (14), and $\hat{s}^T\hat{s} = 0$ (since $\hat{s}^T\hat{s} = 1$) yields an important kinematic identity

$$\hat{S}^T_O \, d\hat{q}_O = dl$$

(15)

While this relation is expressed at point $O$ it is also valid for any arbitrary point.

From (14) and (11) a second important kinematic identity is derived,

$$d\hat{W}_O = \begin{bmatrix} \hat{d}(l\hat{s}) \\ \hat{O}\hat{A} \times \hat{d}(l\hat{s}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \hat{O}\hat{A} \times \hat{O}\hat{A} \times \hat{B}P' \times \hat{B}P' \end{bmatrix} \hat{d}\hat{r}_{P'}$$

(16)

Using the inverse relation of (9) gives the identity,

$$d(l\hat{S}_O) = \hat{M}_O \, d\hat{q}_O$$

(17)

where

$$\hat{M}_O = \begin{bmatrix} 1 \\ \hat{O}\hat{A} \times \hat{O}\hat{A} \times \hat{O}\hat{A} \times \hat{O}\hat{A} \times \hat{B}P' \times \hat{B}P' \times \hat{B}P' \end{bmatrix}$$

(18)
The stiffness matrix maps a differential displacement to the differential load, so from (13),

\[ d\hat{W}_O = d \left( k(l - l_0)\hat{S}_O \right) \]
\[ = k\hat{S}_O dl + k(1 - \rho) l d\hat{S}_O \]
\[ = k\hat{S}_O dl + k(1 - \rho) \left( d(l\hat{S}_O) - dl\hat{S}_O \right) \]  

where

\[ \rho \equiv \frac{l_0}{l} \]  

(20)

Substituting kinematic identities (15) and (17) in (19) yields the desired constitutive relation

\[ d\hat{W}_O = \hat{K}_O \hat{d}_{q_0} \]  

(21)

where \( \hat{K}_O \) is the stiffness matrix represented at \( \hat{O} \) and

\[ \hat{K}_O = k\hat{S}_O \hat{S}_O^T + k(1 - \rho)\hat{M}_O \]  

(22)

Note that the first term in (22) is symmetric while the latter is not. If the spring is unloaded then \( \rho = 1 \), the second term vanishes, and the stiffness matrix becomes symmetric.

4 Skew-Symmetric Properties

The symmetric and skew-symmetric parts of the stiffness matrix are

\[ \hat{K}_{\text{symo}} = \frac{1}{2} \left( \hat{K}_O + \hat{K}_O^T \right) \]
\[ \hat{K}_{\text{skewO}} = \frac{1}{2} \left( \hat{K}_O - \hat{K}_O^T \right) = \frac{1}{2} k(1 - \rho) \left( \hat{M}_O - \hat{M}_O^T \right) \]
\[ = \frac{1}{2} k(1 - \rho) \cdot \begin{pmatrix}
0 & \left( \hat{O} \hat{B}_i - \hat{O} \hat{A}_i \right)^T \\
\left( \hat{O} \hat{B}_i - \hat{O} \hat{A}_i \right)^T & -\left( \hat{O} \hat{A}_i \times \hat{O} \hat{B}_i \times -\hat{O} \hat{B}_i \times \hat{O} \hat{A}_i \right)
\end{pmatrix} \]  

(23)

(24)

The skew-symmetric part is reducible to a simple form. Using \( \hat{O} \hat{B}_i - \hat{O} \hat{A}_i = \hat{A} \hat{B}_i = \hat{l} \hat{h} \) and Jacobi's identity for the lower right hand block gives

\[ \hat{K}_{\text{skewO}} = -\frac{1}{2} k(1 - \rho) \begin{pmatrix}
0 & \hat{A} \hat{B}_i \\
\hat{A} \hat{B}_i & \hat{A} \hat{B}_i \times \hat{A} \hat{B}_i \times
\end{pmatrix} \]
\[ = -\frac{1}{2} k(1 - \rho) \begin{pmatrix}
0 & \hat{l} \hat{h} \\
\hat{l} \hat{h} & \hat{l} \hat{h} \times \hat{A} \hat{B}_i \times \hat{A} \hat{B}_i \times
\end{pmatrix} \]
\[ = -\frac{1}{2} \begin{pmatrix}
0 & k(l - l_0) \hat{h} \times \\
k(l - l_0) \hat{h} \times & (\hat{O} \hat{A}_i \times k(l - l_0) \hat{h} \times)
\end{pmatrix} \]
\[ = -\frac{1}{2} \begin{pmatrix}
0 & \hat{f} \times \\
\hat{f} \times & (\hat{O} \hat{A}_i \times \hat{f} \times)
\end{pmatrix} \]  

(25)

The off-diagonal blocks are skew-symmetric forms of the spring force and the lower right hand block is the skew-symmetric form of the moment of the force about point \( O \). Thus (25) is negative one-half of the applied load in spatial skew-symmetric form

\[ \hat{K}_{\text{skewO}} = -\frac{1}{2} \hat{W}_O \times \]  

(26)

Since there is only one spring, it is also equivalent to one-half of the spring force.

The generalization of the above formulations to \( n \) springs in parallel is simple. Spring related quantities are indicated by the subscript \( i \). The net load on the body is equivalent to the sum of the individual springs forces. The differential of the sum is

\[ d\hat{W}_O = d \left( \sum_i \hat{W}_{Oi} \right) = \sum_i (\hat{K}_Oi \hat{d}_{qO}) \]
\[ = \left( \sum_i \hat{K}_Oi \right) \hat{d}_{qO} \]
\[ = \hat{K}_O \hat{d}_{qO} \]  

(27)

where

\[ \hat{K}_O = \left( \sum_i \hat{K}_Oi \right) \]
\[ \hat{K}_Oi = k_i \rho_i \hat{S}_{Oi} \hat{S}_{Oi}^T + k_i (1 - \rho_i) \hat{M}_i \]
\[ \hat{M}_Oi = \begin{pmatrix}
1 & -\hat{O} \hat{B}_i \times \\
\hat{O} \hat{A}_i \times & -\hat{O} \hat{A}_i \times \hat{O} \hat{B}_i \times
\end{pmatrix} \]  

(28)

(29)

(30)

The symmetric and skew-symmetric parts are

\[ \hat{K}_{\text{symO}} = \sum_i \hat{K}_{\text{symO}i} \]  

(31)

\[ \hat{K}_{\text{skewO}} = \sum_i \hat{K}_{\text{skewO}i} \]
\[ = -\frac{1}{2} \sum_i \hat{W}_{Oi} \times \]
\[ = -\frac{1}{2} \hat{W}_O \times \]  

(32)

(33)

Note that \( \hat{W}_O \) is the external load applied to the body whereas \( -\hat{W}_O \) is the net load applied to the body by the \( n \) springs. The above relation is the central result of this paper and is summarized as the following theorem,
Theorem 1 The skew-symmetric part of the stiffness matrix is one-half the net spring load represented as a spatial cross product operator.

Two corollaries follow as an immediate consequence. At an unloaded equilibrium $\dot{\mathbf{W}}_O = 0$ and it follows from (33) that

**Corollary 2** The stiffness matrix is symmetric if and only if the spring system is at an unloaded equilibrium.

For a $6 \times 6$ matrices, a general one has 36 independent elements, a symmetric one has 21 independent elements, and a skew-symmetric one has 15 independent elements. However, for the class of stiffness matrices under consideration Loncaric (1985) has determined an additional constraint. For a symmetric stiffness matrix the trace of the off-diagonal block vanishes leaving 20 independent parameters. This also applies to the symmetric part of a asymmetric stiffness matrix since the form is similar. From (32), the skew-symmetric part of a stiffness matrix only requires 6 independent elements. Therefore every stiffness matrix (formed from line springs in parallel) only requires a maximum 26 independent elements. It is simple to show that they form a group under addition which combines the stiffnesses of springs in parallel.

**Corollary 3** Stiffness matrices form a subgroup of $6 \times 6$ matrices under the operation of addition with a maximum of 26 independent parameters.

There is also a very simple relationship which connects the stiffness matrix when expressed in the fixed and moving frames and is proved below.

**Corollary 4** The stiffness matrix referenced to the fixed and moving frames at coincident points are transposes.

Combining Corollaries 2 and 4 gives,

**Corollary 5** The fixed and moving frame stiffness matrices are equivalent if and only if the spring system is at an unloaded equilibrium.

Before proving Corollary 4, it is necessary to establish a brief lemma (see also Brand (1947)),

**Lemma 6** For equipollent systems, the derivative of the applied wrench transforms as a spatial vector. The specific case of interest is

$$d\dot{\mathbf{W}}_O = \dot{\mathbf{X}}_{O/p'} \cdot d\dot{\mathbf{W}}_{p'}$$

**Proof of Lemma 6:** Using (1),

$$\dot{\mathbf{W}}_O = \dot{\mathbf{X}}_{O/p'} \cdot \dot{\mathbf{W}}_{p'} = \left[ \frac{d\mathbf{F}}{d\mathbf{F}} \times \mathbf{f} + \mathbf{f}_{p'} \right]$$

$$d\dot{\mathbf{W}}_O = \left[ \frac{d\mathbf{F}}{d\mathbf{F}} \times d\mathbf{f} + d\mathbf{f}_{p'} \right]$$

Transforming the derivatives of $\mathbf{f}$ and $\mathbf{f}_{p'}$ to $\mathbf{F}'$ using (8) and collecting terms gives

$$d\dot{\mathbf{W}}_O = \dot{\mathbf{X}}_{O/p'} \cdot (d\dot{\mathbf{W}}_{p'} + d\mathbf{q}_{p'} \times \dot{\mathbf{W}}_{p'})$$

Using the differentiation formula (7) in the right-hand side establishes (34) and the lemma.

**Proof of Corollary 4.** It is desired to express the stiffness relations with respect to the moving frame $\mathbf{F}'$ at point $p'$ and compare them to the results in the fixed frame $\mathbf{F}$ at $O$. Using the differentiation formula for a moving frame (7) gives

$$d\dot{\mathbf{W}}_{p'} = d\dot{\mathbf{W}}_{O/p'} + \tilde{d}\mathbf{q}_{p'} \times \dot{\mathbf{W}}_{p'}$$

Eliminating in sequence $d\dot{\mathbf{W}}_{p'}$, $\dot{\mathbf{W}}_{p'}$, $d\dot{\mathbf{W}}_{O}$, $d\mathbf{q}_{O}$ using (34), (35), (27), (9) and introducing the transformation equation (55) in the appendix yields after some rearranging,

$$d\dot{\mathbf{W}}_{p'} = \dot{\mathbf{K}}'_{p'} \tilde{d}\mathbf{q}_{p'}$$

where $\dot{\mathbf{K}}'_{p'}$ is given by the congruence transformation

$$\dot{\mathbf{K}}'_{p'} = \dot{\mathbf{X}}_{O/p'}^{-1} (\dot{\mathbf{K}}_{O} + \dot{\mathbf{W}}_{O} \times) \dot{\mathbf{X}}_{O/p'}^{-T}$$

and is the stiffness matrix referenced to the moving frame.

Introducing the symmetric and skew-symmetric parts from (33) yields

$$\dot{\mathbf{K}}'_{p'} = \dot{\mathbf{X}}_{O/p'}^{-1} (\dot{\mathbf{K}}_{sym} + \frac{1}{2} \dot{\mathbf{W}}_{O} \times) \dot{\mathbf{X}}_{O/p'}^{-T}$$

Comparing with (33), the bracketed expression is simply $\dot{\mathbf{K}}_{O}'$ so

$$\dot{\mathbf{K}}'_{p'} = \dot{\mathbf{X}}_{O/p'}^{-1} (\dot{\mathbf{K}}_{O}' \cdot \dot{\mathbf{X}}_{O/p'}^{-T})$$

Finally, although $\dot{\mathbf{X}}_{O/p'}$ is not generally constant, at coincident points it is equal to the identity matrix which establishes the desired result.

$$\dot{\mathbf{K}}'_{p'} = \dot{\mathbf{K}}_{O}'$$

201
5 Numerical Example

An example by Griffis and Duffy (1993) illustrates the relation of the external load to the skew-symmetric portion of the stiffness matrix.

A Stewart platform (see Figure 3) is modelled by two bodies that are connected by six springs in parallel using the following spring stiffnesses $k_1, \ldots, k_6$, free lengths $l_{01}, \ldots, l_{06}$, fixed connection points $a_1, \ldots, a_6$, and moving connection points $b_1, \ldots, b_6$.

$$[k_1 \ldots k_6] = [10 \ 20 \ 30 \ 40 \ 50 \ 60] \quad (44)$$
$$[l_{01} \ldots l_{06}] = [11 \ 12 \ 13 \ 14 \ 15 \ 16] \quad (45)$$
$$[a_1 \ldots a_6] = \begin{bmatrix} 0 & 7 & 3.500 & 3.500 & 0 \\ 0 & 0 & 0.062 & 0.062 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (40)$$

Calculating the spring lengths $l_i$, lines of spring force action $\hat{s}_i$ from (14), and the resulting stiffness matrix $\tilde{K}_O$ from (22) gives

$$[\hat{s}_1 \ldots \hat{s}_6] = \begin{bmatrix} 0.618 & 0.366 & 0.413 & 0.334 & 0.471 & 0.330 \\ 0.333 & 0.417 & 0.089 & -0.244 & -0.146 & 0.248 \\ 0.162 & 0.833 & 0.089 & 0.077 & 0.899 & 0.744 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (49)$$

$$\tilde{K}_O = \begin{bmatrix} 100.00 & 5.21 & 79.54 & 204.94 & 303.54 & -359.63 \\ 5.21 & 39.32 & 5.31 & -581.44 & 8.37 & 516.86 \\ 79.54 & 5.31 & 150.86 & 468.33 & -536.61 & -210.31 \\ -202.48 & 5.31 & -581.16 & 1723.16 & 617.61 & 696.40 \\ -180.24 & 212.01 & -212.21 & -3496.11 & -336.51 & 5383.92 \end{bmatrix} \quad (50)$$

The skew-symmetric part of $\tilde{K}_O$ is

$$\tilde{K}_\text{skew} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -263.86 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (51)$$

The net wrench on the moving body is given by summing the spring forces in (13)

$$\hat{\mathbf{w}}_O = \sum_{i=1}^{6} k_i (l_i - l_{0i}) \hat{s}_{Ui} \quad \begin{bmatrix} 304.653 \\ 59.284 \\ 505.963 \\ 945.102 \\ -2376.391 \\ -369.600 \end{bmatrix} \quad (52)$$

Negative one-half of the external force is

$$-\frac{1}{2} \mathbf{w}_O = \begin{bmatrix} -152.327 \\ -29.642 \\ -252.982 \\ -472.551 \\ 1168.196 \\ 184.803 \end{bmatrix} \quad (53)$$

These values are identical to the ones that form the elements of $\tilde{K}_\text{skew}$. Thus it is shown that $\tilde{K}_\text{skew} = -\frac{1}{2} \mathbf{w}_O \times$ which is the main result of the paper. The calculations were performed using MATLAB and the maximum error in the results between the elements in (51) and (53) are less than $10^{-12}$.

6 Discussion

The principal result is the reduction of the skew-symmetric part of the stiffness matrix to its simplest form. This is the spatial cross product operator representation of negative one-half of the applied load. As an immediate result, it is easily shown that the
stiffness matrix is symmetric if and only if it is at
an unloaded equilibrium. Also the stiffness matrix is
shown to be dependent on a maximum of only 26 pa-
rameters and forms a group under addition. This ana-
lytically proves an observation by Griffis (1993) using
numerical studies that no more than 27 parameters are
necessary. The derivation that the fixed and moving
frame representations of the stiffness matrix are trans-
pose at coincident points generalizes the planar result
given by Pigoski, Griffis, and Duffy (1992). Further,
the transformation in (42) gives a general method to
transform between the fixed and moving representa-
tion at distinct points. This constrains to the well-
known congruence transformation

$$\tilde{K}_{P'} = \tilde{X}_{O/\tilde{P'}}^{-1}(\tilde{K}_O)\tilde{X}_{O/\tilde{P'}}^{-T}$$

(54)

that transforms the fixed frame representation of the
stiffness matrix to another point (that, in this exam-
ple, happens to be in the moving body) and does not
contain the transpose of the stiffness matrix. The
physical difference between the two is that for \(\tilde{K}_{P'}\)
differential changes are viewed from \(\tilde{F}\) while for \(\tilde{K}_P\),
differential changes are observed in \(\tilde{F}'\). In both cases
all quantities are referenced to \(P'\) in \(\tilde{F}'\).

Since this paper dealt with the properties of the
skew-symmetric part it is natural to inquire about the
symmetric part. While some reductions have been
found, they are not so profound as to yield additional
properties or an esthetically appealing representation.
 Apparently, a reason for the significant reduction of
the skew-symmetric part is the close connection with
the Lie algebra of differential changes for a continuous
group of motions. Investigation using the Lie alge-
bra approach of Loncaric (1985) may give additional
physical properties.

Appendix

This section briefly reviews some of the basic prop-
erties of the spatial cross product and its skew-
symmetric operator form. There are several different
representations depending on the selections of ray and
axis coordinates for the factors and product.

Let \(\tilde{a}, \tilde{b}, \text{ and } \tilde{c}\) be spatial vectors in ray coordinates
(e.g. force-like). The spatial cross product is defined as

$$\tilde{a} = \tilde{b} \times \tilde{c} = \begin{bmatrix} \tilde{b} \\ \tilde{b} \times \tilde{c}_O \end{bmatrix} \times \begin{bmatrix} \tilde{c} \\ \tilde{c}_O \end{bmatrix}$$

(55)

where the subscript \(O\) indicates the quantity is rep-
resented at point \(O\). This can also be expressed in an
operator form for \(\tilde{b} \times \tilde{c}\)

$$\tilde{a} = (\tilde{b} \times \tilde{c}) \tilde{e}$$

(56)

where

$$\tilde{b} \times \tilde{c} = \begin{bmatrix} \tilde{b} \times \tilde{c} \\ \tilde{b} \times \tilde{c}_O \end{bmatrix}$$

(57)

and \(\tilde{b} \times, \tilde{b} \times \tilde{c}_O\) are the 3 x 3 skew-symmetric operator
forms for the three-vector cross product. Note that
\(\tilde{b} \times\) is not skew-symmetric but does satisfy relations
analogous to the three-vector cross product such as
Jacobi's identity,

$$(\tilde{b} \times \tilde{c}) \times = \tilde{b} \times \tilde{c} \times -\tilde{c} \times \tilde{b} \times$$

(58)

and rigid body transformations by \(\tilde{X}\)

$$\tilde{X}(\tilde{b} \times \tilde{c}) = (\tilde{X}\tilde{b}) \times (\tilde{X}\tilde{c})$$

(59)

$$(\tilde{X}(\tilde{b} \times \tilde{c})) \times = \tilde{X} ( (\tilde{b} \times \tilde{c}) \times \tilde{X}^{-1}$$

(60)

(See Featherstone (1987) for additional details.)

In this paper the equation under consideration

$$d\tilde{W}_O = \tilde{K}_O \tilde{q}_O$$

$$= (\tilde{K}_{sym} - \tilde{K}_O \times) \tilde{q}_O$$

(61)

has a spatial cross product of the form \(ray = ray \times axis\)
where \(d\tilde{W}_O \text{ and } \tilde{W}_O\) are in ray coordinates and \(d\tilde{q}_O\)
is in axis coordinates. Thus in the above expression
\(\tilde{a} = \tilde{b} \times \tilde{c}\) the spatial cross product operator becomes,

$$\tilde{b} \times = \begin{bmatrix} 0 & \tilde{b} \times \\ \tilde{b} \times & \tilde{b} \times \tilde{b} \times \tilde{b} \times \end{bmatrix}$$

(62)

which is skew-symmetric. Note that identity (58) does
not hold directly unless

$$\tilde{c} \times = \begin{bmatrix} \tilde{c}_O \times & \tilde{c} \times \\ \tilde{c}_O \times & 0 \end{bmatrix}$$

(63)

and \((b \times c) \times\) is defined similar to (57). (For \(a = b \times c\)
there are eight possible combinations of the three spa-
tial cross product operator forms depending on the
selections of ray and axis coordinates for the factors
and the product.) It is direct to show that the trans-
formation laws corresponding to (59) and (60) are

$$\tilde{X}(\tilde{b} \times \tilde{c}) = (\tilde{X}^{-T}\tilde{b}) \times (\tilde{X}\tilde{c})$$

(64)
\[
(\mathbf{X}(b \times \mathbf{e})) \times = \mathbf{X}^{-T} \left( (b \times \mathbf{e}) \times \right) \mathbf{X}^{-1} \quad (65)
\]

Other combinations of ray and axis coordinates can be handled in a similar way but are not needed in this paper.

References


