CENTERS OF STIFFNESS, COMPLIANCE, AND ELASTICITY
IN THE MODELLING OF ROBOTIC SYSTEMS

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Abstract

In the modelling of elastic suspensions between rigid bodies there are three identified points: the centers of elasticity, stiffness, and compliance. Previously, physical properties for the latter two have been unknown. Principal results are: 1) if a compliant axis exists, it must pass through all three centers, and 2) if two compliant axes exist, the three centers coalesce. Additional physical properties that characterize stiffness and the centers are presented. The theory is applied to an RCC device and a dexterous robotic hand.

1 Introduction

Lončarić (1985) defined the center-of-stiffness and the center-of-compliance based on formal considerations. Specifically, at an unloaded equilibrium, an elastic coupling between two rigid bodies can be modelled by a symmetric $6 \times 6$ stiffness or compliance matrix. Lončarić proposed normal forms for the matrices where the off-diagonal $3 \times 3$ blocks are made symmetric by rigid body displacements of the origin. The new origin locations are the centers of stiffness and compliance, depending on whether the stiffness or compliance matrix is normalized. Although the stiffness and compliance matrices are inverses, the two center locations are generally distinct, a curious fact. Other than bringing the matrices to the normal forms, no physical explanations or properties have ever been advanced for the centers of stiffness and compliance.

Dimentberg (1965) introduced the basic modelling of elastic couplings between rigid bodies using $6 \times 6$ matrices and screw theory. He defined three eigenwrenches and three specially defined rotations. The idea was to simplify stiffness analogous to the way principal axes and center-of-mass simplify inertia. For the most part the desired results were found in the simplest cases.

Lipkin and Patterson (1993a) dualized the eigenwrenches of Dimentberg to add three eigentwists. Together these provide a complete decomposition of the matrices into geometric and stationary constitutive properties. The geometric properties are the eigenwrenches and eigentwists (also called wrench- and twist compliant axes) and are used to form a congruence transformation to diagonalize the matrices. The diagonal elements are eigenvalues representing the stationary values of rotational and linear stiffness or compliance. Using the principal kinematic screws of Ball (1900), a center-of-elasticity is defined. It is believed for elastic couplings the eigenwrenches and eigentwists generalize Euler’s principal axes of inertia and the center-of-elasticity generalizes the center-of-mass. These concepts are presented in the following section and used throughout the remainder of the paper.

For robotic applications, Whitney (1982) introduced a compliance center concept to aid insertion tasks. A misalignment of a peg in a hole creates a mo-
ment about the peg. A specially constructed spring device rotated the peg about its tip in the direction of the torque. At the tip is a compliance center since it acts like a three dimensional torsional spring. Latter the device added the ability to translate in the direction of a force at the tip thus simultaneous acting as a three dimensional linear spring.

Rotation about a center in response to a parallel torque and a force through the center producing a parallel translation is the most common characterization of a compliance center. However this is a very special circumstance and does not apply to a general elastic coupling. A further generalization is that of a single compliant axis axis where a rotation about the axis is due to a parallel torque and a force along the axis produces a parallel translation. This is similar to a torsional and linear spring pair. When they exist they can be used to classify elastic couplings, Patterson and Lipkin (1993b). Eigenwrenches and eigentwists are the most complete generalizations and always exist for stable systems in unloaded equilibrium. The center-of-stiffness and -compliance (as defined by Lončaric) and the center-of-elasticity are general concepts that also exist for these systems.

The goal of this paper is to geometrically investigate a general model of elastic couplings. The motivation is to provide a simple characterization in a manner analogous to inertial properties of rigid bodies. However, inertial properties contain only ten independent quantities: one mass, three coordinates for the center-of-mass, three directions for the principal axes, and three principal moments of inertia. A symmetric 6 x 6 stiffness or compliance matrix contains 21 independent parameters. This leads to a much richer geometrical structure and more demanding abstractions than the inertial case.

The primary results are summarized as a series of propositions. Two applications to an RCC device and a dexterous robotic hand inserting a rivet illustrates the response predicted by the theory.

2 Eigenstructure

It is assumed throughout that the elastic system is nonsingular and it is at stable, unloaded equilibrium. The material in this section is required subsequently. Further details are in Lipkin and Patterson (1992a,b).

The fundamental, Cartesian, elastic relations expressed at an arbitrary point O are

\[ \dot{w} = \mathbf{K} \ddot{\mathbf{T}} \]  

(1)

and

\[ \mathbf{T} = \mathbf{C} \dot{\mathbf{w}} \]  

(2)

\[ \dot{\mathbf{w}} = \left[ \begin{array}{c} \ddot{\mathbf{f}} \\ \dddot{\mathbf{T}} \end{array} \right], \quad \mathbf{T} = \left[ \begin{array}{c} \delta \\ \gamma \end{array} \right] \]  

(3)

where \( \dot{\mathbf{w}} \) is the net load given by: \( \ddot{\mathbf{f}} \) the net force, and \( \dddot{\mathbf{T}} \) the net torque about \( O \); \( \mathbf{T} \) is a small motion from equilibrium given by: \( \delta \) a small angular displacement, and \( \gamma \) a small linear displacement of \( O \); \( \mathbf{K} \) and \( \mathbf{C} \) are symmetric 6 x 6 stiffness and the compliance matrices with the 3 x 3 partitions

\[ \mathbf{K} = \left[ \begin{array}{cc} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{D} \end{array} \right], \quad \mathbf{C} = \left[ \begin{array}{cc} \mathbf{E} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{H} \end{array} \right] \]  

(5)

Central to the development in this paper is the geometrical decomposition of the stiffness and compliance matrices. The decomposition arises from solving the following two problems: 1) determine the loads (i.e. eigenwrenches) that translate the system in the direction of the net force and 2) determine the motions (i.e. eigentwists) that produce a pure coupling reaction in the direction of the angular displacement, viz.

\[ \mathbf{C} \left[ \begin{array}{c} \ddot{\mathbf{f}} \\ \dddot{\mathbf{T}} \end{array} \right] = \mathbf{a}_f \left[ \begin{array}{c} \ddot{\mathbf{f}} \\ \mathbf{0} \end{array} \right] \]  

(6)

\[ \mathbf{K} \left[ \begin{array}{c} \delta \\ \gamma \end{array} \right] = \mathbf{k}_r \left[ \begin{array}{c} \mathbf{0} \\ \gamma \end{array} \right] \]  

(7)

where \( \mathbf{a}_f \) is the linear compliance relating the linear displacement to the force and \( \mathbf{k}_r \) is the rotational stiffness relating the couple to the angular displacement. (These problems may also be formulated by minimizing potential energy subject to the respective constraints that 1) the rotational displacement is a unit vector and 2) the force is a unit vector.)

The relations are equivalent to a pair of generalized, singular eigenvalue problems for 6 x 6 matrices. However it is simpler to invert the equations

\[ \mathbf{k}_r \ddot{\mathbf{f}} = \mathbf{K} \ddot{\mathbf{f}} \]  

(8)

\[ \mathbf{a}_f \ddot{\mathbf{f}} = \mathbf{K} \ddot{\mathbf{f}} \]  

(9)

with \( \mathbf{k}_r = \mathbf{a}_f^{-1} \) and \( \mathbf{a}_f = \mathbf{k}_r^{-1} \) and use the partitions (5), so that the first three equations in (8), (9) yield a pair eigenvalue problems for 3 x 3 matrices,

\[ k_f \ddot{\mathbf{f}} = \mathbf{A} \ddot{\mathbf{f}} \]  

(10)
\[ a_\gamma \tilde{\gamma} = H \tilde{\gamma} \]  

(11)

Since the system is assumed to be nonsingular and at a stable, unloaded equilibrium, A and H are symmetric and positive definite. Thus each relation yields three positive eigenvalues that are respectively the stationary (maximum, inflectional, minimum) values of linear stiffness \( k_{j_1} > 0 \) and rotational compliance \( a_{\gamma} > 0 \). Each set of unit eigenvectors \( \tilde{f}_i \) and \( \tilde{\gamma}_i \) form a unique orthogonal set when the eigenvalues are distinct. When there are repeated eigenvalues, an orthogonal set of three linearly independent eigenvectors can always be selected.

Back substitution of the eigenvalues and eigenvectors into (8) and (9) yield respectively three (unit) eigenwrenches \( \tilde{w}_{fi} = [ \tilde{f}_i^T \ \tilde{\gamma}_i^T ]^T \) and three (unit) eigentwists \( \tilde{T}_{\gamma i} = [ \delta_i^T \ \tilde{\gamma}_i^T ]^T \). Since the \( \tilde{f}_i \) are orthogonal, the three eigenwrenches \( \tilde{w}_i \) are along three orthogonal lines that are generally skew. Similarly since the \( \tilde{\gamma}_i \) are orthogonal, the three eigentwists \( \tilde{T}_i \) are along another three orthogonal lines that are generally skew. The two eigensystems form reciprocal three-systems, i.e. they are null spaces of rank three,

\[ \tilde{w}_{fi}^T \tilde{T}_{\gamma j} = 0 \quad i, j = 1, 2, 3 \]  

(12)

These results are used to geometrically decompose the stiffness and compliance matrices using congruence transformations,

\[ K = \begin{bmatrix} f & 0 \\ \tau & \gamma \end{bmatrix} \begin{bmatrix} k_f & 0 \\ 0 & k_\gamma \end{bmatrix} \begin{bmatrix} f & 0 \\ \tau & \gamma \end{bmatrix}^T \]  

(13)

\[ C = \begin{bmatrix} f & \delta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} a_f & 0 \\ 0 & a_\gamma \end{bmatrix} \begin{bmatrix} f & \delta \\ 0 & \gamma \end{bmatrix}^T \]  

(14)

where the \( 3 \times 3 \) blocks are

\[ f = [ \tilde{f}_1 \ \tilde{f}_2 \ \tilde{f}_3 ] \quad \tau = [ \tilde{\tau}_1 \ \tilde{\tau}_2 \ \tilde{\tau}_3 ] \]  

(15)

\[ \gamma = [ \tilde{\gamma}_1 \ \tilde{\gamma}_2 \ \tilde{\gamma}_3 ] \quad \delta = [ \tilde{\delta}_1 \ \tilde{\delta}_2 \ \tilde{\delta}_3 ] \]  

(16)

\[ k_f = \text{diag}( k_{f_1} \ k_{f_2} \ k_{f_3} ) \quad a_f = k_f^{-1} \]  

(17)

\[ k_\gamma = \text{diag}( k_{\gamma_1} \ k_{\gamma_2} \ k_{\gamma_3} ) \quad a_\gamma = k_\gamma^{-1} \]  

(18)

and \( f \) and \( \gamma \) are orthogonal

\[ f^T f = 1 \quad \gamma^T \gamma = 1 \]  

(19)

3 Center-of-Elasticity

As shown by Ball (1900), every three-system has three principal kinematic screws (i.e. wrenches or twists) that intersect orthogonally at a point. For the reciprocal system the principal screws are along the same three axes thus intersecting at the same point. For the three-systems in (12) derived from the elastic model, the common intersection point \( E \) is termed the center-of-elasticity. Linear combinations of the eigenwrenches yields one set of principal screws and linear combinations of the eigentwists yields the other set of principal screws

\[ \begin{bmatrix} m \\ n_\tau \end{bmatrix} = \begin{bmatrix} f \\ \tau \end{bmatrix} \mu_f \]  

(20)

\[ \begin{bmatrix} n_\delta \\ n_\gamma \end{bmatrix} = \begin{bmatrix} \delta \\ \gamma \end{bmatrix} \mu_\gamma \]  

(21)

where the orthogonal directions of the principal screws are

\[ m = [ \tilde{m}_1 \ \tilde{m}_2 \ \tilde{m}_3 ] \]  

(22)

\[ n_\tau, n_\delta \] are \( 3 \times 3 \) matrices, and the linear multipliers \( \mu_f \) and \( \mu_\gamma \) are \( 3 \times 3 \) orthogonal matrices

\[ \mu_f^T \mu_f = 1 \quad \mu_\gamma^T \mu_\gamma = 1 \]  

(23)

since \( f \) and \( \gamma \) represent orthogonal directions. This leads to the following theorem that is used subsequently.

**Theorem 1** An eigenwrench (eigentwist) and a principal screw are equal if and if the axes are parallel.

**Proof:** It is only necessary to prove the eigenwrench case since the eigentwist case is similar. Assume the first eigenwrench and first principal screw are parallel so \( \tilde{f}_1 = \tilde{m}_1 \). Since \( \tilde{f}_1 \) is orthogonal to \( \tilde{f}_2 \) and \( \tilde{f}_3 \) then from (20) the first column of \( \mu_f \) is \( [ 1 \ 0 \ 0 ]^T \) making the first eigenwrench and principal screw equal. The converse is trivial since if the eigenwrench and principal screw are equal then the directions are identical.

The center-of-elasticity also has several important properties of which the first two are proved in Lipkin and Patterson (1992a,b), and used in the sequel.

**Theorem 2** The perpendicular distances from the center-of-elasticity \( E \) to the eigenwrenches (eigentwists) sum to zero,

\[ \sum_i \tilde{p}_{fi} = 0 \quad \left( \sum_i \tilde{p}_{\gamma i} = 0 \right) \]  

(24)

This also implies each set of perpendicular vectors are coplanar (see Figure 1).
Theorem 3 If a compliant axis exists, it passes through the center of elasticity and is coincident with both an eigenwrench and an eigentwist. (Along a compliant axis a force produces a pure translation and a rotation produces a pure couple reaction.)

To investigate the relationship of the center-of-elasticity with other distinguished points, Theorem 2 is generalized in terms of an arbitrary point $O$,

Theorem 4 The vector to $E$ from any point $O$ is equal to one-half the sum of the perpendicular vectors from $O$ to the eigenwrenches (eigentwists), viz.,

$$\vec{r}_E = \frac{1}{2} \sum_i \vec{p}_{fi} \Rightarrow \vec{r}_E = \frac{1}{2} \sum_i \vec{p}_{\gamma i}$$  \hspace{1cm} (25)

Proof: The proof is given only for the eigenwrenches since the proof for the eigentwists is similar. Referring to Figure 2, the position vector $\vec{r}_E = \overrightarrow{OE}$ from some point $O$ to $E$ is

$$\vec{r}_E = \vec{p}_{fi} + \alpha_{fi} \vec{f}_i - \vec{p}_{fi}^E$$  \hspace{1cm} (26)

where $\vec{p}_{fi}$ is the perpendicular distance from $O$ and $\alpha_{fi}$ is the projection of $\vec{r}_E$ on $\vec{f}_i$. Forming the product with $\vec{f}_i^T$ gives

$$\alpha_{fi} = \vec{f}_i^T \vec{r}_E$$  \hspace{1cm} (27)

and since the $\vec{f}_i$ are orthonormal

$$\sum_i \alpha_{fi} \vec{f}_i = \vec{r}_E$$  \hspace{1cm} (28)

Summing (26) over index $i$

$$3\vec{r}_E = \sum_i \vec{p}_{fi} + \sum_i \alpha_{fi} \vec{f}_i - \sum_i \vec{p}_{fi}^E$$  \hspace{1cm} (29)

and using (24) and (28) yields the desired result (25).

4 Centers-of-Stiffness and Compliance

In the following it is assumed that all quantities that are origin dependent are with respect to origin $O$ unless otherwise noted.

Let $\vec{r}_S = \overrightarrow{OS}$ be the position vector to center-of-stiffness $S$ and $\vec{r}_C = \overrightarrow{OC}$ be the position vector to the center-of-compliance. Representation of the stiffness at $S$ and the compliance at $C$ is given by the congruence transformations

$$K_S = \begin{bmatrix} 1 & \vec{r}_S \times \vec{f}_i \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{bmatrix} 1 & \vec{r}_S \times \vec{f}_i \end{bmatrix}$$  \hspace{1cm} (30)

$$C_C = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \begin{bmatrix} E & G \\ G^T & H \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$  \hspace{1cm} (31)

For normal forms of the matrices, the off-diagonal blocks must themselves be symmetric

$$B + A\vec{r}_S \times = B^T - \vec{r}_S \times A$$  \hspace{1cm} (32)

$$G - \vec{r}_C \times H = G^T + H\vec{r}_C \times$$  \hspace{1cm} (33)

Letting

$$\vec{b} \times = \frac{1}{2}(B - B^T)$$  \hspace{1cm} (34)
\[ \ddot{\mathbf{x}} = \frac{1}{2}(\mathbf{G} - \mathbf{G}^T) \]  

(35)

gives

\[ 2\ddot{\mathbf{b}} = -(\mathbf{A}\ddot{\mathbf{r}}_S + \ddot{\mathbf{r}}_S \times \mathbf{A}) \]  

(36)

\[ 2\ddot{\mathbf{e}} = \mathbf{H}\ddot{\mathbf{r}}_C + \ddot{\mathbf{r}}_C \times \mathbf{H} \]  

(37)

The expressions are equivalent to

\[ [\mathbf{A} - \text{tr}(\mathbf{A})\mathbf{1}]\ddot{\mathbf{r}}_S = 2\ddot{\mathbf{b}} \]  

(38)

\[ [\text{tr}(\mathbf{H})\mathbf{1} - \mathbf{H}]\ddot{\mathbf{r}}_C = 2\ddot{\mathbf{e}} \]  

(39)

These solutions are given by Lončarić (1985) and a tensorial solution is presented in the Appendix. For stable systems the matrices are invertible.

(Note if a matrix in (38) or (39) is singular then at least one eigenvalue is negative or two are zero making the system either unstable or singular. If the rank of the matrix is two then the centers form a line, if the rank is one they form a plane, and if the rank is zero they encompass all points.)

A physical interpretation of these equations is developed by expressing \( \ddot{\mathbf{b}} \) and \( \ddot{\mathbf{e}} \) using the decompositions (13), (14) and the partitions (5)

\[ \mathbf{B} = \mathbf{f}_i\tau_i^T = \sum_i k_{ji}\ddot{\mathbf{r}}_i\dot{\tau}_i^T \]  

(40)

\[ \mathbf{G} = \delta_{ji}\tau_i^T = \sum_i a_{ji}\ddot{\mathbf{r}}_i\dot{\tau}_i^T \]  

(41)

so that

\[ 2\ddot{\mathbf{b}} = \mathbf{B} - \mathbf{B}^T = \sum_i k_{ji}(\ddot{\mathbf{r}}_i\dot{\tau}_i^T - \ddot{\mathbf{r}}_i\dot{\tau}_i^T) \]  

(42)

\[ 2\ddot{\mathbf{e}} = \mathbf{G} - \mathbf{G}^T = \sum_i a_{ji}(\ddot{\mathbf{r}}_i\dot{\tau}_i^T - \ddot{\mathbf{r}}_i\dot{\tau}_i^T) \]  

(43)

Using the vector identity

\[ \dddot{\mathbf{v}}^T - \dddot{\mathbf{v}}^T = (\dddot{\mathbf{x}} \times \mathbf{u}) \times \]  

(44)

yields

\[ 2\ddot{\mathbf{b}} = \sum_i k_{ji}(\dddot{\mathbf{r}}_i \times \dddot{\mathbf{r}}_i) \]  

(45)

\[ 2\ddot{\mathbf{e}} = \sum_i a_{ji}(\dddot{\mathbf{r}}_i \times \dddot{\mathbf{r}}_i) \]  

(46)

However the perpendicular vectors from \( \mathbf{O} \) to the eigenwrenches and eigentwists are (for example see Brand (1947)),

\[ \ddot{\mathbf{p}}_{ji} = \frac{\dddot{\mathbf{r}}_i \times \dddot{\mathbf{r}}_i}{\dddot{\mathbf{r}}_i \cdot \dddot{\mathbf{r}}_i} \]  

(47)

\[ \ddot{p}_{ji} = \frac{\dddot{r}_i \times \dddot{r}_i}{\dddot{r}_i \cdot \dddot{r}_i} \]  

(48)

Since \( \dddot{r}_i \) and \( \dddot{r}_i \) are unit vectors then (45) and (46) become

\[ \ddot{\mathbf{b}} = -\frac{1}{2} \sum_i k_{ji}\dddot{\mathbf{r}}_i \]  

(49)

\[ \ddot{\mathbf{e}} = \frac{1}{2} \sum_i a_{ji}\dddot{\mathbf{r}}_i \]  

(50)

These expressions are very similar to ones for the center-of-elasticity (25) of Theorem 2 except that they involve weighted sums of the perpendicular vectors. Eliminating \( \ddot{\mathbf{b}} \) and \( \ddot{\mathbf{e}} \) using (38) and (39) gives

\[ [\text{tr}(\mathbf{A})\mathbf{1} - \mathbf{A}]\ddot{\mathbf{r}}_S = \sum_i k_{ji}\dddot{\mathbf{r}}_i \]  

(51)

\[ [\text{tr}(\mathbf{H})\mathbf{1} - \mathbf{H}]\ddot{\mathbf{r}}_C = \sum_i a_{ji}\dddot{\mathbf{r}}_i \]  

(52)

By moving \( \mathbf{O} \) to \( S \) (\( C \)) then \( \ddot{\mathbf{r}}_S = \ddot{\mathbf{O}} (\ddot{\mathbf{r}}_C = \ddot{\mathbf{O}}) \) and the following theorem is proved,

**Theorem 5** At the center-of-stiffness \( S \) (center-of-compliance \( C \)) the perpendicular vectors to the eigenwrenches (eigentwists) weighted by the stationary values of linear stiffness (rotational compliance) sum to zero, viz.

\[ \sum_i k_{ji}\dddot{\mathbf{r}}_i = 0 \quad \left( \sum_i a_{ji}\dddot{\mathbf{r}}_i = 0 \right) \]  

(53)

where the \( \dddot{\mathbf{p}}_{ji}^S \) (\( \dddot{\mathbf{p}}_{ji}^C \)) are perpendicular vectors from \( S \) (\( C \)) to the axes of the eigenwrenches (eigentwists).

Since the weighted perpendicular vectors sum to zero then as an immediate corollary,

**Corollary 6** The perpendicular vectors from the center-of-stiffness \( S \) (center-of-compliance \( C \)) to the eigenwrenches (eigentwists) are coplanar.

There is a strong similarity between the relations for the center-of-elasticity in (24) and those for the centers-of-stiffness and -compliance in (53). The difference is that the relation for the \( E \) is purely geometric whereas the relations for \( S \) and \( C \) involve geometric quantities weighted by the stationary values of constitutive properties. With this view, note that the decompositions (13) and (14) based on the eigenvalue problems decouple the stiffness and compliance matrices into purely geometric quantities (the eigenvectors) and purely constitutive quantities (the eigenvalues).
It is also useful to solve (51) and (52) for the locations of $S$ and $C$. Note that under the congruence transformations (30) and (31) that $A$ and $H$ are invariant. Using the decompositions (13), (14) and the partitions (5)

$$A = f k_f f^T$$

$$H = \gamma a_\gamma \gamma^T$$

and using the orthogonality relations (19) yields

$$[tr(k_f)1 - k_f](f^T f_s) = f^T \sum_i k_f i \bar p_{f i}$$

$$[tr(a_\gamma)1 - a_\gamma](\gamma^T f_C) = \gamma^T \sum_i a_{\gamma i} \bar p_{\gamma i}$$

(As an aside, these equations are useful for examining the cases where the matrix is not invertible. The terms in the parentheses represent the projections of the position vectors into the directions of the respective eigenvectors. Since each matrix in the brackets is diagonal then, if the rank is two the line of centers is in the direction of an eigenvector, and if the rank is one then the plane of centers is normal to an eigenvector.) Completing the inversions gives,

$$\tilde r_S = f [tr(k_f)1 - k_f]^{-1} f^T \sum_i k_f i \bar p_{f i}$$

$$\tilde r_C = \gamma [tr(a_\gamma)1 - a_\gamma]^{-1} \gamma^T \sum_i a_{\gamma i} \bar p_{\gamma i}$$

5 Compliant Axes

This section establishes the fundamental relationships between the three centers, Ball's principal screws, the eigenwrenches and eigentwists, and the existence of compliant axes. The results are summarized in four propositions.

Respectively, from (24) and (53) the following relations can be directly derived,

$$\bar p_{f 1}^E = \alpha_2 \bar f_2 - \alpha_3 \bar f_3$$

$$\bar p_{f 2}^E = \alpha_3 \bar f_3 - \alpha_1 \bar f_1$$

$$\bar p_{f 3}^E = \alpha_1 \bar f_1 - \alpha_2 \bar f_2$$

and

$$k_{f 1} \bar p_{f 1}^S = \beta_2 \bar f_2 - \beta_3 \bar f_3$$

$$k_{f 2} \bar p_{f 2}^S = \beta_3 \bar f_3 - \beta_1 \bar f_1$$

$$k_{f 3} \bar p_{f 3}^S = \beta_1 \bar f_1 - \beta_2 \bar f_2$$

The coefficients are just the projections of $\bar p_{f i}^E$ and $k_{f i} \bar p_{f i}^S$ in the three orthogonal directions of $f$. Expressing (25) at $S$ instead of $O$ and (58) at $E$ instead of $O$ yields two expressions for the vectors between $E$ and $S$

$$\tilde r_E^S = \frac{1}{2} \sum_i \bar p_{f i}^S$$

$$\tilde r_E^C = \frac{1}{2} \sum_i \bar p_{f i}^C$$

where $\tilde r_E^C = -\tilde r_E^S$. Substitution of (60) into (63) and (61) into (62) yields

$$\tilde r_{E}^S = k_{f 3} - k_{f 2} \alpha_1 \bar f_1 + k_{f 1} - k_{f 3} \alpha_2 \bar f_2$$

$$k_{f 3} - k_{f 2} \alpha_1 \bar f_1 + k_{f 1} + k_{f 3} \alpha_2 \bar f_2$$

$$k_{f 1} - k_{f 3} \alpha_3 \bar f_3$$

$$k_{f 2} + k_{f 1} \alpha_3 \bar f_3$$

$$\tilde r_{E}^S = \frac{1}{2} \left( (a_{f 3} - a_{f 2}) \beta_1 \bar f_1 + (a_{f 3} - a_{f 1}) \beta_2 \bar f_2$$

$$+ (a_{f 2} - a_{f 1}) \beta_3 \bar f_3 \right)$$

Similarly for the eigentwists,

$$\bar p_{\gamma 1}^E = \eta_2 \bar \gamma_2 - \eta_3 \bar \gamma_3$$

$$\bar p_{\gamma 2}^E = \eta_3 \bar \gamma_3 - \eta_1 \bar \gamma_1$$

$$\bar p_{\gamma 3}^E = \eta_1 \bar \gamma_1 - \eta_2 \bar \gamma_2$$

and

$$a_{\gamma 1} \bar p_{\gamma 1}^C = \xi_2 \bar \gamma_2 - \xi_3 \bar \gamma_3$$

$$a_{\gamma 2} \bar p_{\gamma 2}^C = \xi_3 \bar \gamma_3 - \xi_1 \bar \gamma_1$$

$$a_{\gamma 3} \bar p_{\gamma 3}^C = \xi_1 \bar \gamma_1 - \xi_2 \bar \gamma_2$$

The coefficients are just the projections of $\bar p_{\gamma i}^E$ and $a_{\gamma i} \bar p_{\gamma i}^C$ in the three orthogonal directions of $\gamma$. Expressing (25) at $C$ instead of $O$ and (59) at $E$ instead of $O$ yields two expressions for the vectors between $E$ and $C$

$$\tilde r_E^C = \frac{1}{2} \sum_i \bar p_{\gamma i}^C$$

$$\tilde r_E^C = \gamma [tr(a_\gamma)1 - a_\gamma]^{-1} \gamma^T \sum_i a_{\gamma i} \bar p_{\gamma i}^E$$

where $\tilde r_E^C = -\tilde r_E^C$. Substitution of (66) into (69) and (67) into (68) yields

$$\tilde r_{E}^C = \frac{a_{\gamma 3} - a_{\gamma 2}}{a_{\gamma 3} + a_{\gamma 2}} \eta_1 \bar f_1 + \frac{a_{\gamma 1} - a_{\gamma 2}}{a_{\gamma 1} + a_{\gamma 3}} \eta_2 \bar f_2$$

$$+ \frac{a_{\gamma 2} - a_{\gamma 1}}{a_{\gamma 2} + a_{\gamma 1}} \eta_3 \bar f_3$$

$$\tilde r_{E}^C = \frac{1}{2} \left( (k_{\gamma 3} - k_{\gamma 2}) \xi_1 \bar f_1 + (k_{\gamma 1} - k_{\gamma 3}) \xi_2 \bar f_2$$

$$+ (k_{\gamma 2} - k_{\gamma 1}) \xi_3 \bar f_3 \right)$$

The preceding results lead to the following theorems,
Theorem 7 An eigenwrench (eigen-twist) passes through E if and only if it also passes through S (C).

Proof: The proof is for the eigenwrench case. Assume the 3rd eigenwrench passes through E so \( \mathbf{\tilde{f}}_{E} = \mathbf{0} \). From (60)

\[
\alpha_1 = \alpha_2 = 0
\]

(72)

and substitution into (64) yields

\[
\mathbf{\tilde{f}}_{E}^S = \frac{k_{f_2} - k_{f_1}}{k_{f_2} + k_{f_1}} \alpha_3 \mathbf{\tilde{f}}_3
\]

(73)

Thus the 3rd eigenwrench axis intersects S since it is in the direction from E to S. For the converse, assume the 3rd eigenwrench axis passes through S so \( \mathbf{\tilde{f}}_{E}^S = \mathbf{0} \). From (61)

\[
\beta_1 = \beta_2 = 0
\]

(74)

and substitution into (65) yields

\[
\mathbf{\tilde{f}}_{E}^S = \frac{1}{2} (a_{f_2} - a_{f_1}) \beta_3 \mathbf{\tilde{f}}_3
\]

(75)

Thus the 3rd eigenwrench axis intersects E since it is in the direction from S to E.

Theorem 8 E and S (C) are coincident if and only if the eigenwrenches (eigentwists) are principal screws.

Proof: The proof is for the eigenwrenches. First assume that the eigenwrenches are principal screws and thus intersect at E. From (60) the \( \alpha_i = 0 \) so from (64) \( \mathbf{\tilde{f}}_{E}^S = \mathbf{0} \) making E and S coincident. The converse considers three distinct cases with \( \mathbf{\tilde{f}}_{E}^S = \mathbf{0} \).

i) Let the \( k_{f_1} \) be distinct. From (62) the \( \alpha_i = 0 \) so from (62) the eigenwrenches pass through E and are thus principal screws. ii) Let \( k_{f_1} = k_{f_2} \neq k_{f_3} \). From (64) \( \alpha_1 = \alpha_2 = 0 \) and from (60) \( \mathbf{\tilde{f}}_{E}^S = \mathbf{0} \) so that the third eigenwrench passes through E and is thus a principal screw. Since now \( \mathbf{\tilde{f}}_3 = \mathbf{\tilde{m}}_3 \) then from (20) the vector directions \( \mathbf{\tilde{f}}_1, \mathbf{\tilde{f}}_2 \) are linear combinations of \( \mathbf{\tilde{m}}_1, \mathbf{\tilde{m}}_2 \). However since the eigenvalues \( k_{f_1}, k_{f_2} \) have a repeated value then from (10) \( \mathbf{\tilde{f}}_1, \mathbf{\tilde{f}}_2 \) are not unique and any pair of orthogonal directions normal to \( \mathbf{\tilde{f}}_3 \) may be selected. In particular, select \( \mathbf{\tilde{f}}_1, \mathbf{\tilde{f}}_2 \) parallel to \( \mathbf{\tilde{m}}_1, \mathbf{\tilde{m}}_2 \) so that from Theorem 1 the eigenwrenches are also principal screws. iii) Let \( k_{f_1} = k_{f_2} = k_{f_3} \). Since there is only one distinct eigenvalue then from (10) the directions of the eigenvectors \( \mathbf{\tilde{f}}_1 \) are arbitrary. Thus selecting them parallel to the \( \mathbf{\tilde{m}}_1 \) makes the eigenwrenches principal screws.

Theorem 9 A compliant axis passes through all three centers E, S and C.


Figure 3: Schematic of an RCC Device, Drake (1977).

Proof: Let the pair \( \mathbf{\tilde{w}}_{f_1}, \mathbf{\tilde{t}}_{n} \) be a compliant axis. By Theorem 2 it must pass through E. By Theorem 6, \( \mathbf{\tilde{w}}_{f_1} \) must also pass through S and \( \mathbf{\tilde{t}}_{n} \) must also pass through C. But, since their axes are, by definition, collinear they all pass through E, S and C.

Since compliant axes are along orthogonal eigenwrenches (or eigentwists) then, as an immediate consequence of the above theorem,

Corollary 10 If two compliant axes exist then E, S, and C coalesce.

6 Applications

The presented analysis gives additional tools to understand and predict the properties of stiffness. The first application uses the compliance matrix measured for the a Remote Center of Compliance (RCC) device. There is good agreement with the intended design and the existence of a compliance center is confirmed. It also illustrates Corollary 10 where the three centers coalesce. A second application is to a dexterous hand grasping a rivet for hole insertion. The existence of a single compliant axis illustrates Theorem 9 where the centers become collinear. However, the remaining eigenwrenches and eigentwists do not all facilitate the insertion process and may lead to wedging or jamming.

Example 1. Whitney (1982) explains the development and properties of the RCC device. The calibration of the device is detailed in Drake (1977) and a schematic is shown in Figure 3. One linkage provides a lateral compliance and one provides a rota-
tional compliance. Using a numerically controlled machine, three translations and three rotations about the $x$, $y$, and $z$ axes were applied to the tip of the device at the expected compliance center. The resulting six wrenches were measured by a six degrees-of-freedom force/torque cell attached to the RCC base. The wrench and twist data are listed in Drake (1977) and used to form a $6 \times 6$ matrix equation to solve for the compliance,

$$
\mathbf{C} = \begin{bmatrix}
0.0898 & 0.0058 & 0.0007 & -0.0092 & -0.1937 & -0.1958 \\
-0.0054 & 0.0034 & -0.0004 & 0.2385 & -0.0614 & 0.1956 \\
-0.0001 & 0.0004 & -0.0007 & 0.0047 & 0.0034 & 0.0914 \\
-0.0074 & 0.1681 & 0.0023 & 2.8346 & -0.1127 & 0.7191 \\
-0.1685 & -0.0380 & 0.0009 & -0.4246 & 2.4148 & 2.4551 \\
-0.0122 & -0.2149 & -0.0260 & -1.6734 & 0.0816 & 130.9437
\end{bmatrix}
$$

(76)

where the units of force, length, and angle are newtons, millimeters, and milliradians. Note that the resulting matrix is asymmetric. The author ascribes this to measurement error. Using the symmetric part of $\mathbf{C}$ results in the following stationary values of linear stiffness $k_{fi}$ and angular stiffness $k_{gi}$

$$
k_{fi} = \text{diag}(13.2 \ 13.4 \ 1498.1)
$$

(77)

$$
k_{gi} = \text{diag}(0.4387 \ 0.4090 \ 0.0076)
$$

(78)

In Figure 4 the eigenwrenches (solid lines) and eigentwists (dotted lines) are shown. They all intersect at a point that is approximately on the $z$ axis 0.0791 mm above the origin where the centers of elasticity, stiffness, and compliance coalesce. There is a compliant axis in the $z$ direction since an eigenwrench and an eigentwist are coincident along the $z$ axis.

The remaining two eigenwrenches are nearly in the $x$ and $y$ directions with zero pitches indicating that they are forces. The eigentwists are rotated about 30.7° and have zero pitches indicating that they are pure rotations. However, note that the two linear stiffnesses and two rotational stiffness in the $x - y$ plane are nearly equal. If each pair had identical stiffnesses then there would be compliant axes in all $x - y$ directions and the centers would coalesce exactly on the $z$ axis. While this is not the actual case, the RCC device comes very close to this goal, as the design intends. Axes through the centers but not in the $x - y$ plane are not compliant axes since the stiffnesses in the $z$ direction are different. For example a force in the $x - z$ plane will cause a translation in a different direction of the $x - z$ plane.

In this example only the symmetric part of the compliance matrix was used and the asymmetry was associated with measurement error. As a test, the asymmetric stiffness matrix was computed and then made symmetric. The results were very similar to the ones obtained by making the compliance matrix symmetric. For example, the centers are located at 0.0795 mm on the $z$ axis and the maximum error in the stiffness ($z$ angular) is less than 4%. This would tend to support the claim that the asymmetry was primarily due to artifacts in the data.

Example 2. Stiffness properties are important for controlling dexterous robotic hands. Cutkosky and Kao (1989) consider using two fingers from the Stanford/JPL hand to manipulate a 0.02 m (20 mm) rivet for insertion, see Figure 5. Each finger has a "soft" fingertip that transmits forces in three directions through the contact and a moment along the normal to the tangent contact plane. This is sufficient to provide full force closure for the hand and rivet. Using representative compliance values measured for the cables, joints, links, and fingertips and including the effect of the servo system, the combined structural and servo stiffness matrix for the grasped rivet at the tip is given as

$$
\mathbf{K} = \begin{bmatrix}
2490 & 0 & 0 & 0 & 258 & 0 \\
0 & 28900 & 0 & 191 & 0 & 0 \\
0 & 0 & 61610 & 0 & 0 & 0 \\
0 & 191 & 0 & 0 & 22 & 0 \\
258 & 0 & 0 & 0 & 37 & 0 \\
0 & 0 & 0 & 0 & 0 & 35
\end{bmatrix}
$$

(79)

where the units of force is newtons, length is meters, and angle is radians. The stationary values of stiffness

Figure 4: Eigenwrenches (solid lines) and eigentwists (dotted lines) for the RCC device. There is a pair along the $z$ axis forming a compliant axis and the three centers coalesce.
are all distinct

\[ k_f = \text{diag}(2490, 28900, 61610) \]  \hspace{1cm} (80)

\[ k_r = \text{diag}(10.268, 20.738, 35.000) \]  \hspace{1cm} (81)

Figure 6 shows the eigenwrenches (solid lines), eigentwists (dotted lines), and the three centers. All eigenwrenches and eigentwists are along the coordinate directions. They have zero pitches indicating pure forces and rotations respectively. The \( z \) axis contains a collinear eigenwrench-eigentwist pair that indicates a compliant axis. From the symmetry of Figure 5 this is reasonable.

All three centers lie on the compliant axis as predicted in Theorem 9. Point \( E \) is 0.0485m (48.5mm) above the origin and respectively \( S \) is at 0.0021m (2.1mm) and \( C \) is at 0.0671m (67.1mm). Since \( E \)

[Figure 6: Eigenwrenches (solid lines) and eigentwists (dotted lines) for the a dexterous hand. There is a pair along the \( z \) axis forming a compliant axis and the three centers are collinear.]

is a geometric center it occupies a symmetrical position with respect to the eigenwrenches and eigentwists. Points \( S \) and \( C \) are also located between the eigenwrenches and eigentwists since their distances from the origin are calculated as sums of the perpendicular distances with positive weights.

The eigentwist near the tip is useful for inserting the rivet since a torque in the \( z \) direction will create a corrective rotation about the \( z \)-axis to null out the torque similar to an RCC device. However for the eigentwist in the \( y \) direction, a \( y \) torque will create a rotation about a parallel axis 0.1102m (110.2mm) from the tip. This may lead to jamming or wedging of the rivet in the hole. Similarly the eigenwrench near the tip in the \( y \) direction will correctively translate the rivet away from a force along the axis. However, the only \( z \) forces that will purely translate the rivet in the \( z \) direction must be applied along the eigenwrench also 0.1102m (110.2mm) from the tip. To determine the motion in response to an \( z \) force at the tip, transform the force to the \( z \) direction eigenwrench and include the moment about the \(-y\) direction. The force now produces a corrective translation, however the moment now creates a negative rotation about the \( y \) direction eigentwist. This is an undesirable effect that could lead to wedging or jamming.
7 Closing Remarks

Close connections have been established amongst the centers-of-elasticity, stiffness, and compliance and the existence of compliant axes. Most importantly, a compliant axis passes through all three centers. The centers-of-stiffness and compliance have been found to have geometric properties whereas previously their existence was based on algebraic formalisms. The complex 21 dimensional parameter space of the elastic model can be better understood by developing properties that can be represented by simpler geometric entities. This is the case with the 10 dimensional inertia space of a rigid body where the center-of-mass and the principal axes are fundamental concepts. However, for the elastic case the physical abstractions are more demanding as a result of the greater number of free parameters.

The goal of this paper has been to present theory for detailing physical abstractions of elastic systems and to show its usefulness for the analysis of practical and realistic devices. Two applications to an RCC device and a dexterous robotic hand predicted physical behaviors. While the RCC analysis mostly confirmed known properties, the analysis of the robotic hand explained desirable and undesirable characteristics for rivet insertion. This information could be used to redesign the hand, grasp, or servo system to provide a more satisfactory response though beyond the scope of this paper.

Appendix

To solve equations (41) and (42) consider the matrix equation

$$\bar{\varepsilon} \times = \bar{r} \times \mathbf{H} + \mathbf{H} \bar{r} \times$$  \hspace{1cm} (82)

In tensor notation

$$\bar{\varepsilon} = [\varepsilon_{i}], \bar{r} = [r_{i}], \mathbf{H} = [h_{ij}]$$  \hspace{1cm} (83)

and

$$\bar{r} \times = [-\varepsilon_{ijk} r_{k}]$$  \hspace{1cm} (84)

where $\varepsilon_{ijk}$ is the permutation tensor so (82) becomes

$$\varepsilon_{ijk} e_{k} = \varepsilon_{ipq} r_{q} h_{pj} + h_{ip} \varepsilon_{pq} r_{q}$$  \hspace{1cm} (85)

Multiplying both sides of (85) by $\varepsilon_{ijm}$ and using the identity

$$\varepsilon_{ipq} \varepsilon_{imn} = \delta_{pm} \delta_{qn} - \delta_{pn} \delta_{qm}$$  \hspace{1cm} (86)

where $\delta_{ij}$ is the identity tensor yields

$$\varepsilon_{ijm} \varepsilon_{ijk} e_{k} = [\varepsilon_{ijm} \varepsilon_{ipq} h_{pj} + \varepsilon_{ijm} \varepsilon_{pq} h_{ip}] r_{q}$$  \hspace{1cm} (87)

Also, $\varepsilon_{ijm} = \varepsilon_{jmi}$ and $\varepsilon_{pq} = \varepsilon_{qp}$, therefore

$$(\delta_{jj} \delta_{km} - \delta_{jm} \delta_{jk}) e_{k} = [\delta_{jp} \delta_{qm} - \delta_{jq} \delta_{mp}] h_{pj}$$

$$+ (\delta_{mq} \delta_{ip} - \delta_{mp} \delta_{iq}) h_{ip} r_{q}$$  \hspace{1cm} (88)

Further,

$$\delta_{ij} g_{i...} = g_{i...}, \text{ and, } \delta_{ii} = n,$$  \hspace{1cm} (89)

where $n$ is the number of all possible choices for $i$ or the dimension of the space over which $\delta_{ij}$ is defined. In this case $n = 3$ so

$$e_{m} = [h_{ii} \delta_{mq} - \frac{1}{2} (h_{mq} + h_{qm})] r_{q}$$  \hspace{1cm} (90)

which corresponds to the matrix equation

$$\bar{\varepsilon} = [\text{tr}(\mathbf{H})1 - \frac{1}{2}(\mathbf{H} + \mathbf{H}^{T})] \bar{r}$$  \hspace{1cm} (91)

References

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