Enumeration of Singular Configurations for Robotic Manipulators

Kinematic singularities are important considerations in the design and control of robotic manipulators. For six degree-of-freedom manipulators, the vanishing of the determinant of the Jacobian yields the conditions for the primary singularities. Examining the vanishing of the minors of the Jacobian yields further singularities which are special cases of the primary ones. A systematic procedure is presented to efficiently enumerate all possible singular configurations. Special geometries of representative manipulators are exploited by expressing the Jacobian in terms of vector elements. In contrast to using a joint-angle space approach, the resulting expressions yield direct physical interpretations.

Introduction

The Jacobian $J$ of a manipulator is a linear transformation which maps joint velocities to instantaneous end-effector velocity, and thus serves as a conversion from joint space to Euclidean end-effector space. The inverse problem of determining the joint velocities is solvable only when the desired end-effector motion is in the column space $J$. Assuming six or less joints, then if the columns of $J$ are linearly dependent, the manipulator is said to be in a special or singular configuration. In this paper, the terms "special configuration" and "singular configuration" are used interchangeably. For a six-jointed robot in such a configuration, the columns of the Jacobian no longer span a six-space and the manipulator end-effector loses one or more degrees of motion freedom. If the desired motion of the end-effector is not in the space spanned by the Jacobian, the linearly dependent joint velocities will approach infinity. Since attempting infinite joint velocities will (at best) saturate any robot's actuators, the actual path near a singularity will diverge from the planned one. It is very important, therefore, to identify special configurations in order to either avoid them or to employ specially devised algorithms for controlling the manipulator near singularities. A clear understanding of the types of possible singularities and their degenerate cases is necessary for the development of satisfactory control algorithms. Previous studies of mechanism and manipulator singularities include Hunt (1978), Paul and Stevenson (1983), Lai and Yang (1986), Wang and Waldron (1986), and Soylu and Duffy (1988).

Special configurations fall into two categories: uncertainty configurations and stationary configurations, Lipkin and Duffy (1985). In an uncertainty configuration, the solution to one or more joint angels in the inverse kinematics analysis is mathematically indeterminate. It is then possible for the dependent joints to move through a finite displacement while the end-effector is held fixed. If only infinitesimal joint movement is possible with the end-effector fixed, then the robot is in a stationary configuration. A sufficient condition for a manipulator to be in a stationary configuration is for its end-effector to be on the boundary of its workspace. Note that the definitions for uncertainty and stationary configurations differ from those put forth in Hunt (1978) and Sugimoto, Duffy, and Hunt (1982), but are similar to the concepts of finite and infinitesimal internal degrees-of-freedom as put forth in Lai and Yang (1986).

Although manipulator singularities are generally regarded as undesirable, they do have some potential benefits. In an uncertainty configuration, the manipulator gains internal degrees-of-freedom. This mobility might be used to allow the inner links to avoid obstacles. Another benefit of singularities is as follows. As $J$ becomes singular, its columns, which are screw quantities, no longer span six-degrees-of-freedom and the rank of $J$ decreases by at least one. For each degree of motion freedom lost, there exists one reciprocal screw which, if applied as a wrench to the end-effector, produces no virtual work at the manipulator joints. This implies that all of the load is handled by the joint constraints, Wang and Waldron (1986). Furthermore, as a manipulator approaches a singularity, effects of the singularity begin to appear, Paul and Stevenson (1983). If the manipulator is only approximately at a singular configuration, the torque demands on the actuators will be small. Effectively, the mechanical advantage of the manipulator increases in the proximity of a singular configuration. Another important aspect of singularities is that they occur when the manipulator changes branches. Most industrial manipulators restrict branching abilities via hardware or software limits which diminish the robot's capabilities. A thorough understanding of singularities may enable an automatic and optimal selection of branches.

A redundant manipulator, i.e., one with more than six degrees-of-freedom, generally has an infinite number of configurations possible for a given end-effector location. Thus it is possible to optimize some performance criteria. For example, subchains in the manipulator may be in singular con-
figurations in order to avoid obstacles or to decrease torque loads. Therefore, the deliberate use of manipulator singularities may be critical for the optimal control and operation of such robots.

As the causes and effects of manipulator singularities become better known, they may be used in the design of future manipulators. An analogy to control theory serves as an illustration. The poles or "ill-behaved" points of a control system are analyzed and then placed for optimal system behavior in the design phase. Likewise, as the theory behind robot singularities advances, more design effort may be applied to synthesizing singular configurations to produce more efficient manipulator behavior.

This paper describes a method of enumerating all singularities or joint dependencies which exist for a given manipulator and includes extensive examples with representative industrial manipulators. The analysis is performed by expressing the Jacobian in end-effector space using vector quantities (Pieper, 1968, Whitney, 1972). Previous algorithms for analyzing singular configurations via a general Jacobian are performed in joint space (i.e., describing singularity conditions in terms of joint angles), e.g., Waldron, Wang, and Bolin, (1984), Litvin and Castelli (1985), Soylu and Duffy, (1988). It should be emphasized that it is the method of finding singularities which is new, since many of the singularities which emerge from the examples have been seen before. However, by using a Cartesian vector approach, a great deal of physical insight into the singularity problem is gained, and it is possible to further identify degeneracies of the primary singular configurations. By contrast, the joint-angle approach tends to obscure geometric intuition, since the analysis is done in an abstract configuration space. Further, the joint-space method does not directly address the problem of determining special cases of the primary singular configurations. Understanding these subcases is necessary in an effort to better understand manipulator singularities. As far as the authors are aware, a complete enumeration of the singularities for a manipulator has not appeared previously in the literature. Additionally, the concepts of active and passive joints are expanded upon, Hunt (1985), and a theorem regarding the relationship between invariant screw systems and passive joints is similarly broadened.

**Primary Singular Configurations**

The Jacobian transformation from joint motion to end-effector motion may be expressed as,

$$\begin{bmatrix} \omega' \\ v \\ \omega_N \end{bmatrix} = J(\theta) \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix}$$

(1)

where $N$ is the number of manipulator joints, $v$ and $\omega$ describe the translational velocity and angular velocity of the end-effector respectively, $\theta$ is the vector of joint angles, and $(\omega_1 \ldots \omega_N)$ are the joint velocities. The Jacobian may be expressed by

$$J = \begin{bmatrix} s_1 & \ldots & s_N \\ r_1 \times s_1 & \ldots & r_N \times s_N \end{bmatrix}$$

(2)

(Whitney, 1972) where $s_i$ is the unit vector along joint axis $i$ and $r_i$ is the position vector from the origin to any point on joint axis $i$.

The search for special configurations of six degree-of-freedom manipulators begins by setting $|J| = 0$. If $N < 6$, the Jacobian is nonsquare and other methods, such as using the Gram determinant, e.g., Soylu and Duffy (1988), or evaluating minors may be used to determine singularity conditions. Since the determinant of a square matrix is invariant with respect to changes in a coordinate system, the origin used for the evaluation of the Jacobian may be placed to facilitate the calculation. The end-effector may be conceptually extended to include a coordinate origin anywhere on or off the robot. Virtually all industrial robot designs include parallel or perpendicular joints, intersecting joints, and link lengths or joint offsets of zero, which make the analysis much simpler than for a general six degree-of-freedom manipulator. If the robot has three intersecting joints, the intersection point is an obvious choice for the placement of the origin, since the three $r$ vectors associated with the three co-intersecting axes become zero (Lipkin and Duffy, 1982).

In order to illustrate the concepts in this paper, it is useful to refer to an example which is examined in detail. The MBA manipulator is a prototype industrial robot which is representative of the special geometries used in industrial robots. Later, other examples are examined and are used to amplify the discussion. Figure 1 illustrates the six revolute joint manipulator with joints 2 and 3 parallel, and joints 4, 5, 6, co-intersecting at a point to form a wrist. The joint offsets $S_{22}$, $S_{33}$, and $S_{55}$ are zero, as are the link lengths $a_{14}$, $a_{45}$, and $a_{56}$. All twist angles are either 0 or $\pm \pi/2$. By placing the coordinate origin at the wrist point, the Jacobian simplifies to

$$J = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ r_1 \times s_1 & r_2 \times s_2 & r_3 \times s_3 & 0 & 0 & 0 \end{bmatrix}$$

(3)

Setting the determinant to zero and simplifying yields,

$$|J| = l_1 r_1 \times s_1 - l_2 r_2 \times s_2 - l_3 r_3 \times s_3 - l_4 s_4 s_5 s_6 = 0.$$  

(4)

Using a vector identity, the first factor may be expressed as,

$$l_1 r_1 s_1 s_2 - l_2 r_2 s_2 s_3 - l_3 r_3 s_3 s_4 - l_4 s_4 s_5 s_6 = 0.$$  

(5)

Since $s_1$ and $s_3$ are parallel the last determinant in (5) vanishes and thus

$$|J| = l_1 r_1 s_1 s_5 - l_2 r_2 s_2 s_3 - l_3 s_3 s_4 s_6 = 0.$$  

(6)

Setting each of the three factors in (6) to zero yields the three primary singularity conditions for the manipulators. A primary singular configuration is defined as the most general configuration which causes a single factor of the Jacobian determinant to vanish. (It is useful to note that this form of $|J| = 0$ is also applicable to the Puma robot). It is desired to examine the conditions by which each factor vanishes while the other factors remain nonzero.

In general there are three distinct ways in which a vector triple product (i.e., a $3 \times 3$ determinant) may vanish:

(a) Any one of the vectors may vanish identically.

(b) Any two vectors may be parallel with (a) being satisfied.

![Fig. 1 The MBA manipulator](image-url)
(c) The three vectors may be coplanar without (a) or (b) being satisfied. This provides a systematic method of determining all possible primary singularities by directly utilizing easily understood vector quantities. Simultaneously, the vector method provides direct insight into the physical configurations which produce the singularities.

The first factor in (6),

\[ |r_1s_1s_2| = 0 \]  \quad (7a)

vanishes for \( r_1 = 0 \). Geometrically, this means that the wrist joint is concurrent with joint 1 [Fig. 2(a)]. Since the four screws \( s_1, s_4, s_5, s_6 \) representing joints 1, 4, 5, and 6 are concurrent, they form a linearly dependent set of rank 3. This is termed the \([1456]_3\) configuration, where the outside subscript represents the rank of the bracketed system. Note that in the general \([1456]_3\) configuration, no subset of two or three joints is dependent. This is a minimally dependent set, also called a minimal set, which is defined as a set of screws which are linearly dependent, with the general stipulation that no subset of screws are dependent. This is an uncertainty configuration since the four joints form a spherical four-bar mechanism and are free to rotate when the end-effector is fixed.

The second term in (6),

\[ |r_2r_3s_3| = 0 \]  \quad (7b)

vanishes when \( r_2, r_3, \) and \( s_3 \) are coplanar, implying that \( r_2 \) and \( r_3 \) are parallel. This in turn implies that joint 3 is fully extended [Fig. 3(a)]. The joint dependence of this configuration can be explained in the following way. Since the last three joints intersect, they simulate the behavior of a spherical pair, allowing a rotation of the end-effector about any line through the wrist point. When joint 3 is fully extended, the last three joints can create an instantaneous rotation which is coplanar with and parallel to joints 2 and 3. Therefore, the five screws \( s_3, s_4, s_5, s_6 \) become linearly dependent, and the minimal set is referred to as \([23456]_4\). Since joint 3 is fully extended, it precludes any finite motion of the joints when the end-effector is held fixed. This is therefore a stationary configuration.

If the third determinant in (6) vanishes,

\[ |s_4s_5s_6| = 0 \]  \quad (7c)

joints 4, 5 and 6 are coplanar. This is possible only when joints 4 and 6 are collinear, and it is therefore termed the \([46]_1\) case (Fig. 4). Joints 4 and 6 are free to rotate about their common axis without changing the end-effector location and thus \([46]_1\) is an uncertainty configuration.

In almost all previous singularities analyses, general formulations for singularity conditions are expressed in joint space [exceptions are Lipkin (1983), Lipkin and Duffy (1982, 1985) and Paden (1985)]. In terms of joint angles, an expression for the Jacobian is

\[ |J| = a_{21}S_{44}[a_{12} + a_{23}c_2 + S_{44}s_2 + 1](c_1)s_3 \]  \quad (8)

Fig. 2 The [1456]_3 primary singularity and its subcases

Fig. 3 The [23456]_4 primary singularity and its subcases
Soylu and Duffy (1988), where \( c_i \) and \( s_i \) are the cosine and sine of joint angle \( \theta_i \), and the subscript \( i+j \) refers to \( \theta_i + \theta_j \). Equations (6) and (8) each have three factors which identify the primary singular configurations:

\[
\begin{align*}
1_{r_1} s_{s_1} s_{s_1} & = 0 \quad (a_{13} + a_{23} c_2 + S_{46} s_5 + 1) = 0 \\
1_{r_2} s_{s_2} s_{s_2} & = 0 \quad c_1 = 0 \\
1_{s_3} s_{s_3} s_{s_3} & = 0 \quad s_3 = 0.
\end{align*}
\]

Although the joint space expressions in (9b) and (9c) have simple physical meanings, the interpretation of (9a) is difficult. This is in contrast to the vector expression which is easily interpreted. A similar situation occurs for examining the singularity subcases in both Cartesian vector form and joint space form.

**Active and Passive Joints**

Once in a special configuration, only the actuation of certain joints can return the Jacobian to full rank. Joints which are able to take a manipulator out of a special configuration are termed active joints, while joints without this ability are termed naturally passive joints. Obviously, the first and last joints of a manipulator are always passive since they cannot change the relative geometry (position and orientation) of the joints. Table 1 lists the active joints for the three primary singularity configurations of the MBA. These follow directly from the joint space expressions in (9).

Passive joints can affect joint dependencies once in a special configuration, causing degeneracies of the primary singular cases. The definitions for active and passive joints may be modified to apply to portions of the Jacobian. If a minimal set of \( m \) dependent columns is selected from the Jacobian, it is often possible, by actuation of the proper joints, to increase or reduce the number of columns in the minimally dependent set. For example, the [46] \(_1\) dependency [Fig. 2(c)] may be converted to the [1456] \(_3\) dependency [Fig. 2(a)] by rotating joint 4. Joint 4 therefore is active in the sense that it is converts the dependency between joints 1, 4, and 6 to a larger minimal set ([1456],). A joint which is passive with respect to a minimal set of joints is one which cannot alter the number of joints in the minimal set. Otherwise a joint is said to be active with respect to a minimal set of joints.

**Invariant Screw Systems**

An invariant screw system is one in which the space spanned by the system does not change with finite displacements of its joints, see Hunt (1978) and Hunt (1986). Examples of invariant systems include spherical pairs and cylindrical pairs. A theorem concerning invariant screw systems is now presented, which is an extension of a theorem put forth in Soylu and Duffy (1988).

**Theorem 1.** All of the joints in an invariant system at the end of a manipulator chain are passive.

**Proof.** Consider a manipulator chain consisting of two subsystems: subsystem 1 which is an invariant screw system, and subsystem 2 which is non-invariant. If subsystem 1 contains \( m \) degrees-of-freedom, then a basis for the system may consist of \( m \) screws which are constant with respect to the link that attaches to subsystem 2. If subsystem 2 contains \( n \) degrees-of-freedom, its basis consists of \( n \) configuration-dependent screws. Thus a basis for the complete manipulator chain may be expressed by \( (\dot{S}_1, \ldots, \dot{S}_{n+m}) \) (the primes indicate that the screws forming the basis are not necessarily the same as the joint screws). In general, actuating the joints in subsystem 2 changes its basis screws. If any screws become linearly dependent, the rank of the Jacobian is reduced. By definition, however, the screws forming the basis of subsystem 1 do not change, and as a group cannot reduce in rank. Furthermore, since the invariant system is at the end of the manipulator chain, actuating the joints in the invariant system do not change the relative geometry between the joints in the non-invariant system. Therefore, it follows that all of the joints in the invariant system are passive. Q.E.D.

Many industrial robots employ kinematic subchains which simulate invariant systems except at certain special configurations. For example, three non-coplanar intersecting revolute joints form an equivalent spherical pair. However, when the three joints become coplanar, the three-system degenerates to a two-system. This is exactly what happens in the [46] \(_1\) singular configuration, except that joints 4 and 6 also become collinear. Similarly, three parallel revolute joints span the three degrees-of-freedom possible in planar motion except when they become coplanar. Systems which always span the same space, except for a finite number of special configurations, are referred to here as quasi-invariant. Quasi-invariant screw systems contain all of the properties of true invariant systems except at singular configurations. Therefore, Theorem 1 may be qualified for manipulators with quasi-invariant screw systems.

**Theorem 2.** If a quasi-invariant screw system is at one end of a manipulator, then all joints in the system are passive unless the quasi-invariant system itself is in a special configuration. In other words, if singular configurations are deleted from the quasi-invariant system, then the previous theorem applies.

Since joints 4, 5, and 6 of the MBA manipulator constitute a quasi-invariant screw system, the joints are passive unless joints 4 and 6 are collinear. When this occurs joint 5 becomes active. Since joint 1 is connected to ground and thus always passive, it follows that joints 2, 3, and 5 are the only active joints with respect to the [1456], primary singularity (see Table 1).

**Enumeration of Singularities**

After determining the most general configurations in which the Jacobian is singular, the degeneracies of the primary singular cases may be examined. Each primary case contains a linear dependence among 2 to \( N \) joint axes, where \( N \) is the number of joints in the manipulator. However, dependencies among subsets of the dependent joint sets occur frequently. For every set of \( m \) dependent vectors in a special configuration,
tion, \( m > 2 \), there exists the possibility of 2 to \( m - 1 \) joints in the set becoming dependent for specific manipulator configurations. The most efficient way to search for degeneracies is a recursive algorithm, whereby all sets of \( m - 1 \) vectors in each set of \( m \) joints is examined for dependencies, down to sets of two vectors. The search pattern therefore takes on a tree-like structure and Fig. 5 illustrates the case for the MBA robot. The total number of possible dependent-joint sets for a six degree-of-freedom robot is thus

\[ 6C_5 + 6C_4 + 6C_3 + 6C_2 = 57. \]  

(10)

where \( 6C_n \) is the number of combinations of \( n \) items taken \( m \) at a time. In practice, it is not necessary to search all 57 cases and subcases for dependencies. If any particular set of joints is found to be linearly independent, then any subset will be independent and need not be examined.

There are a number of ways to determine the conditions for linear dependence in a nonsquare matrix of manipulator joint screws. The most satisfactory method found is to evaluate all of the largest possible minors or subdeterminants. If all minors are zero, the matrix has less than full rank. This method is somewhat time-consuming, but is not difficult to perform and reliably finds all singular manipulator configurations. Note that the method of minors may also be used on nonsquare Jacobians. As in the full Jacobian analysis, the amount of algebra necessary to evaluate the minors may be minimized by placing the coordinate origin at a convenient position and orientation. In addition, the fact that the manipulator is already in a singular configuration can be exploited to greatly simplify the analysis. This is the preferred method used in the paper.

Another method is to express the Jacobian in joint coordinates and proceed with the minor evaluation. However, as discussed earlier, geometrical insight may be lost using this method, and it is often difficult to interpret the joint-space equations that result. Still another method of determining singular configurations is to examine the Gram determinant of the joint set, using either Cartesian vectors or joint angles. The Gram determinant of a vector set \( \{ \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n \} \) is defined as

\[ G(\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n) = \det([\mathbf{S}_1 \mathbf{S}_2 \ldots \mathbf{S}_n]^T [\mathbf{S}_1 \mathbf{S}_2 \ldots \mathbf{S}_n]) \]  

(11)

The Gram determinant vanishes if and only if the screws are linearly dependent. Although this single scalar condition appears attractive, there are some important disadvantages to the method. First, the Gram determinant is identically equivalent to the sum of the squares of the \( 6C_N \) \( N \times N \) minors of the matrix. For example, see Hadley (1961). This doubles the order of the individual minors and thus increases the complexity of the physical interpretation of the singularities. Second, this introduces extraneous solutions which correspond to imaginary singularities or increase the multiplicity of the real singularities. Third, the Gram determinant is generally noninvariant with respect to translations in coordinate system.

Different choices of the coordinate system introduce different extraneous solutions.

A simple example illustrates how the method of minors may greatly reduce in complexity while a robot is in a singular configuration. Primary case [1456], is composed of four screws, \( \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4 \). The three-screw subgroups of [1456] are [145], [146], [156], and [456]. Each must be examined to determine if it is a minimally dependent set. The analysis of the \( \{ \mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_4 \} \) set for dependence is detailed. Placing the origin at the wrist joint so that \( i = \mathbf{S}_4 \) and \( j = \mathbf{S}_3 \) (Fig. 2), then

\[ \{ \mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_4 \} = \begin{bmatrix} s_1 & i & j \\ 0 & 0 & 0 \end{bmatrix}. \]  

(12)

Thus a necessary and sufficient condition for \( \{ \mathbf{S}_1, \mathbf{S}_3, \mathbf{S}_4 \} \) to be linearly dependent is \( i_1 s_1 s_3 s_4 = 0 \). This occurs if \( s_1 = s_4 \) are parallel, \( s_1 = s_3 \) are parallel, or \( s_1, s_3, s_4 \) are coplanar.

The last case is illustrated in Fig. 2(b). It is important to note that the wrist is not assumed to be concurrent with joint 1, i.e., the [1456], singularity is not occurring, then this condition emerges from the minor evaluation of \( \{ \mathbf{S}_3, \mathbf{S}_4, \mathbf{S}_5 \} \). However, by exploiting the geometric conditions of the primary singular configurations, the analysis is simplified considerably. The minor-evaluation process is similar for subcases [146], [156], and [456].

Subcases [145], [146], and [156], [Figs. 2(b) and 2(c)] are formed from [1456], by actuating joint 4, while subcase [156], occurs only when \( s_1 \) and \( s_3 \) are perpendicular [Fig. 2(d)] and requires the actuation of joints 2, 3, and 5. Case [456] is not a minimally dependent set, but occurs only when joints 4 and 6 become dependent.

Continuing down the tree, two-joint subsets of [1456], are examined. Joint pairs [14], [15], and [16], occur and are shown in Figs. 2(e) and 2(g). [14], may also be reached directly from [1456], by actuating joints 2 and 3. In addition, [46], is valid since it is a primary singularity.

Five possible rank 3 subcases arise from basic case [23456], and two exist as minimal sets, [2346], and [2356], [Fig. 3(b) and 3(c)]. Two minimal subcases of rank 2 exist. [236], is a subset of [2346], and is achieved by actuating joint 5 [Fig. 3(e)]; and [235], is a direct subset of [23456], and requires the actuation of joint 4 [Fig. 3(d)]. The only rank 1 subspace of [23456] which is not contingent on special manipulator dimensions is [46]. The [25], and [26], subcases exist only if \( a_3 = a_4 \). The parameters \( a_3 \) and \( a_4 \) are called active dimensions, since their values affect the rank of the Jacobian. Table 2 lists the minimal sets of vector dependencies which exist for the manipulator.

### Singularity Combinations

Once all the primary singular configurations and their subcases are enumerated, combinations of two or more singularities may be examined. If two or more singularities exist simultaneously, they frequently have the effect of reducing the rank of the Jacobian further. It is very important to identify when a manipulator is in a combination singularity, since it may require the actuation of two or more joints to escape from such a configuration. Figures 6(a)–6(n) illustrate con-
figurations which result from all possible combinations of the singularities enumerated in the tree. A single configuration may arise from several different combinations. Table 3 describes the possible combinations and configurations which produce them. For purposes of clarity, combinations involving dimension-dependent singular configurations have been omitted. It is interesting to note from Table 3 that the minimum rank of $|J|$ is four. This is true even when the three primary singularities are occurring simultaneously as in Fig. 6(l).

Additional Examples

Singularity analyses are outlined for two common industrial robots, the Puma and the Cincinnati-Millacron T3. For brevity, enumeration of the singularity combinations and most

Table 2  Joint dependencies for the MBA manipulator

<table>
<thead>
<tr>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
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<tbody>
<tr>
<td>[145]₂</td>
<td>[2346]₃</td>
<td></td>
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<tr>
<td>[146]₂</td>
<td>[2356]₃</td>
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<tr>
<td>[156]₂</td>
<td>[2456]₃</td>
<td>*</td>
</tr>
<tr>
<td>[14]₁</td>
<td>[235]₂</td>
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<tr>
<td>[16]₁</td>
<td>[246]₂ *</td>
<td>*</td>
</tr>
<tr>
<td>[46]₁</td>
<td>[256]₂ *</td>
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</table>

*—denotes dimension-dependent sets

Table 3  Singularity combinations for the MBA manipulator

| Combination          | Rank of $|J|$ | Figure |
|----------------------|-------------|--------|
| [1456]₃ × [23456]₄  | 5           | 6a     |
| [1456]₃ × [2346]₃   | 5           | 6b     |
| [1456]₃ × [235]₂    | 5           | 6c     |
| [1456]₃ × [236]₂    | 5           | 6d     |
| [145]₂ × [23456]₄   | 5           | 6b     |
| [145]₂ × [2346]₃    | 5           | 6b     |
| [145]₂ × [235]₂     | 4           | 6d     |
| [146]₂ × [23456]₄   | 5           | 6c     |
| [146]₂ × [235]₂     | 4           | 6c     |
| [16]₁ × [235]₂      | 4           | 6e     |
| [16]₁ × [46]₁       | 5           | 6f     |
| [14]₁ × [46]₁       | 5           | 6g     |
| [14]₁ × [46]₁       | 4           | 6d     |
| [16]₁ × [46]₁       | 4           | 6i     |
| [23456]₄ × [46]₁    | 5           | 6j     |
| [2346]₃ × [46]₁    | 4           | 6j     |
| [235]₂ × [46]₁      | 4           | 6k     |
| [1456]₃ × [23456]₄ + [46]₁ | 4 | 6m |
| [1456]₃ × [2346]₃ + [46]₁ | 4 | 6m |
| [1456]₃ × [235]₂ + [46]₁ | 4 | 6m |
| [145]₂ × [23456]₄ + [46]₁ | 4 | 6m |
| [145]₂ × [2346]₃ + [46]₁ | 4 | 6m |
| [146]₂ × [235]₂ + [46]₁ | 4 | 6n |

Fig. 6  Singularity combinations for the MBA
Figures of the singular configurations are omitted for the Puma and T³, but are presented in Pohl (1988).

The Puma robot is very similar in geometry to the MBA except that the link length $a_{12} = 0$ and joint offset $s_{22} \neq 0$. As with the MBA, the Jacobian is formulated with the coordinate origin at the wrist. Because of this similarity the Jacobian expressions (2)-(6) are identical, as are the conditions for the primary singularities,

$$|J| = |s_{1}r_{1}s_{2}| |r_{1}r_{2}s_{3}| |s_{4}s_{5}s_{6}| = 0.$$  \hspace{1cm} (14)

One singularity condition emerges as each of the three determinants in (14) vanishes. The second and third factors yield the primary cases $[23456]_4$ and $[46]_1$, which are identical to primary singularities found on the MBA. The first case, $|s_{1}r_{1}s_{2}| = 0$, is satisfied when $s_{1}$, $s_{2}$, and $r_{1}$ are coplanar. This occurs when the wrist lies in the plane formed by $s_{1}$ and $s_{2}$. In this configuration, a linear combination of $S_{4}$, $S_{5}$, and $S_{6}$ can produce a rotation of the end-effector $(s_{46})$ which co-intersects and is coplanar with $S_{1}$ and $S_{2}$ (Fig. 7). Therefore, this is a $[12456]_4$ configuration.

Figure 8 shows the joint-depency tree for the Puma. The rank 3 subcases of $[23456]_4$ are $[12345]_4$, $[12456]_4$, and $[12456]_4$. In the first three cases, a linear combination of $S_{4}$ and $S_{6}$ form a vector which is coplanar and concurrent with the other two vectors in the set. The $[2456]_3$ subcase exists when the four joints are concurrent at a point, which happens only if $a_{12} = S_{46}$. In addition, there are four rank two subcases stemming from the $[12456]_4$ primary singularity, and three rank one cases. Table 4 lists the eighteen joint dependencies for the Puma, fourteen of which are unique. Note that Groups 2 and 3 are the same for the Puma and the MBA.

The Cincinnati-Millacron T³ industrial robot is a six revolute joint manipulator with joints 2, 3, and 4 parallel. The joint offsets $s_{23}$, $S_{34}$, $S_{45}$, and $s_{56}$ are zero, as are the link lengths $a_{12}$ and $a_{46}$. All twist angles are either 0 or $\pm \pi / 2$. The main geometric feature of the T³ which will be exploited is the series of three parallel joint axes [Fig. 9(a)]. Placing the coordinate origin at the intersection of joints 1 and 2 with i along $s_{1}$ and j along $s_{2}$, the Jacobian becomes

$$J = \begin{bmatrix}
    s_{1} & s_{2} & s_{3} & s_{4} & s_{5} & s_{6} \\
    0 & 0 & r_{3} & r_{4} & r_{5} & r_{6} & s_{46} 
\end{bmatrix}.$$  \hspace{1cm} (15)

After simplifying the determinant, it may be expressed as the product of three factors,

$$|J| = |s_{1}r_{1}s_{2}| |s_{3}r_{3}s_{4}| |s_{5}r_{5}s_{6}|.$$  \hspace{1cm} (16)

The three singularity conditions which arise from the three factors in (16) are as follows. For $|s_{1}r_{1}s_{2}| = 0$, $s_{1}$ and $r_{1}$ are parallel, and the intersection of joints 5 and 6 lies in the plane formed by joints 1 and 2. This forms a dependency among all six joints $([123456]_4)$ and an uncertainty configuration [Fig. 9(a)]. Joints 1, 5, and 6 can form a rotation parallel to joints 2, 3, and 4 to simulate a planar four-bar mechanism. For $|s_{3}r_{3}s_{4}| = 0$, $s_{3}$, $s_{4}$, and $s_{5}$ are coplanar. Joints 2, 3, 4, and 6 are parallel and form a planar four-bar linkage and an uncertainty configuration [Fig. 9(b)]. For $|s_{5}r_{5}s_{6}| = 0$, $r_{3}$ and $r_{6}$ are parallel since joint 3 is fully extended. Joints 2, 3, and 4 are coplanar and therefore dependent. Since links $a_{23}$ and $a_{24}$ are fully stretched, this is a stationary configuration [Fig. 9(c)]. Table 5 summarizes the possible joint dependencies found for the T³, which are also represented by the tree in Fig. 10.

The $[123456]_4$ primary case contains three rank 4 subsets, $[12356]_4$, $[12356]_4$, and $[13456]_4$. Sets $[12356]_4$ and $[12456]_4$ contain the same lower subcases, $[1256]_4$, $[156]_4$, $[15]_4$, and $[16]_4$. Primary case $[12456]_4$ contains four dependent sets with three screws each: $[234]_4$ (itself a primary case), $[236]_4$, $[246]_4$, and $[256]_4$. Rank 1 subsets of $[2346]_3$, are $[24]_1$, $[26]_1$, and $[46]_1$, although only $[26]_1$ is not dependent on special manipulator dimensions.
Conclusion

The identification of singularity configurations is very important in the design and control of industrial manipulators. This paper presents a straightforward and effective method for determining all singular configurations which exist for a given manipulator, with examples using industrial robots presented throughout. The evaluation of the Jacobian and determination of joint dependencies are done in terms of vector quantities instead of the more traditional joint angles approach. The vector approach provides a clean geometric interpretation of the singularity conditions by directly exploiting special manipulator geometries. This also facilitates the determination of subcases of the primary singular configurations by the method of minors. Passive and active joints are discussed, and a theorem concerning invariant screw systems and passive joints is broadened.

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Table 5 Joint dependencies for the \( \tau^3 \) manipulator

<table>
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<th>Group 2</th>
<th>Group 3</th>
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<td>[123456]_5</td>
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<td>[256]_2</td>
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*—denotes dimension-dependent sets

References


