STRUCTURE OF ROBOT COMPLIANCE

Timothy Patterson
Engineering Science and Mechanics
Harvey Lipkin
The George W. Woodruff School of Mechanical Engineering
Georgia Institute of Technology
Atlanta, Georgia

Abstract An alternative treatment of robot compliance is developed by applying screw theory to the compliance matrix eigenvalue problem. The compliance and stiffness matrix eigenvalue problems are shown to be equivalent. The eigenscrews are demonstrated to be Ball's (1900) principal screws of the potential. Several new propositions are presented characterizing the compliance matrix eigenstructure. In a companion paper, Patterson and Lipkin (1990) the results are used to classify general compliance matrices. Using a novel formulation, a second eigenvalue problem is developed. It is used to generalize the wrench-compliant axes of Dimentberg (1965) to include twist-compliant axes. Together these two types of compliant axes are shown to define conjugate screw systems of the potential.

There are however, relatively limited previous investigations into the structure of robot compliance matrices and related subjects, Ball (1900), Dimentberg (1965) and Loncaric (1985). The motivation for this investigation comes primarily from these latter works.

Ball (1900) examines the dynamics of rigid bodies using screw theory. Although, this at first appears unrelated, some interesting concepts are presented. Among these, is the idea of principal screws of inertia. An impulsive wrench applied to a quiescent rigid body about a principal screw of inertia produces an instantaneous twist on the same screw. Ball shows that a body with $n$ degrees of freedom has $n$ principal screws of inertia. These principal screws of inertia are coreciprocal. For an unconstrained body the principal screws of inertia are collinear with the principal moment of inertia axes. Ball also applies the same ideas to a rigid body in a potential field. This leads to the concept of principal screws of the potential. A rigid body in a potential field, subjected to a twist deformation on a principal screw of the potential produces a wrench on the same screw. A body with $n$ degrees of freedom in a potential field has $n$ principal screws of the potential. Ball finds the principal screws of the potential using an eigenvalue formulation. His investigation of this eigenvalue problem is limited to showing that the eigenvectors, or principal screws of the potential, are coreciprocal and real.

Dimentberg (1965) applies screw theory to the statics and small vibrations of an elastically suspended rigid body. The structure of the elastic suspension for such a body is characterized by a special system of six screws. These screws correspond to wrenches produced by three orthogonal translations and three orthogonal rotations. The system has a "center-of-elasticity" only when the six screws form a pair of coincident orthogonal trihedrals.

Loncaric (1985) examines the compliance matrices of robotic mechanisms using Lie Algebra. A different definition of center-of-compliance is presented. This
definition is based on using rotations and translations to diagonalize the off-diagonal blocks of the 6x6 compliance matrix. Using this definition, the "center-of-compliance" and the "center-of-elasticity" are generally not at the same point. In the process of developing this definition, Loncaric shows that the compliance matrix cannot be diagonalized by rigid body translations and rotations. In this investigation a payload/manipulator system is modeled as an unconstrained, elastically supported rigid body, i.e. a rigid body in a potential field (see Figure 1). This is the same model discussed by Ball, Dimentberg, and Loncaric. The compliant characteristics of such a system are given by a symmetric 6x6 compliance matrix. From Ball's work, it is known that the axes of the principal screws of inertia for a rigid body are found by solving the moment of inertia eigenvalue problem. Similarly, the principal screws of the potential for an elastic system may be found by solving the compliance matrix eigenvalue problem. There is however, an important difference between the two cases. The principal screws of inertia are always orthogonal, as the 6x6 inertia matrix can always be diagonalized by simple coordinate system rotations and translations. This is not the case with the compliance matrix. Understanding the compliance matrix structure requires a determination of the relationships that exist between its eigenscrews, the principal screws of the potential.

The remainder of the paper is organized in three sections. Section 2 contains a development of the compliance matrix eigenvalue problem. It presents solutions to the problem in both axis and ray Plücker coordinates, see Lipkin and Duffy (1988), Springer (1964).

![Figure 1. Elastically Suspended Body.](image)

The equivalence of the compliance matrix eigenvalue problem and the corresponding stiffness (or elastic) matrix eigenvalue problem is also demonstrated. Section 3 proves a number of important propositions which characterize the eigenstructure of the compliance matrix. In Section 4 a new type of eigenvalue problem is introduced. This results in the general concept of wrench- and twist-compliant axes. Several propositions concerning the characteristics of the wrench- and twist-compliant axes are presented.

This paper makes two major contributions. First, insight into the properties of the compliance matrix structure is gained by the application of screw theory to the compliance eigenvalue problem. The properties of the compliance matrix are stated in terms of several new propositions. Second, using a new eigenvalue formulation the wrench-compliant axes of Dimentberg are generalized to include the twist-compliant axes. Together, these two types of compliant axes are shown to define a system of conjugate screws of the potential.

2. The Compliance Matrix Eigenvalue Problem

A general model for a payload/manipulator system is an unconstrained, elastically suspended rigid body (see Figure 1). A wrench applied to such an elastic system produces a change in body position or a twist deformation. As the body twists the elastic suspension deforms, thus the phrase twist deformation. The term twist usually refers to the combined translational and rotational time rates of change in position. It is retained here to emphasize that change in position is the result of both a translation and a rotation. From the opposite point of view, an applied twist deformation causes the the elastic suspension to exert a counteracting wrench. The relationship between the wrench and twist is described by a symmetric 6x6 compliance matrix in the first case, and by a symmetric 6x6 stiffness matrix in the second case. The two matrices provide equivalent descriptions of the system elastic characteristics. Following Dimentberg (1965), a derivation of the stiffness matrix is included in Appendix A.

The compliance matrix eigenvalue problem must be formulated in a coherent fashion if the results are to be meaningful. It is possible to do this using either ray or axis screw coordinates. Stated in terms of compliance, the relationship between applied wrenches and twist deformations is

$$T = C \mathbf{w}$$

(2.1)

where,

$$T = \begin{bmatrix} \delta^T & \gamma^T \end{bmatrix}^T, \quad w = \begin{bmatrix} f^T & \tau^T \end{bmatrix}^T,$$

and C is the compliance matrix. The term T is a twist deformation in axis coordinates. Its two 3x1 vectors components are, linear deformation $\delta$, and rotational deformation $\gamma$. The term w is a wrench in ray coordinates. Its two 3x1 vectors components are, force $f$, and torque $\tau$. Notice, that the wrench and the twist deformation are stated in different coordinates. The formulation of the compliance matrix eigenvalue problem requires that both quantities be stated in the same coordinates. Among other things, this preserves the integrity of the units and ensures that the results are independent of the coordinate frame. If the twist and wrench are not stated in the same coordinates the results are not invariant and thus become meaningless.

Using the the operator $\Delta$, the twist is stated in ray coordinates,

$$\Delta \mathbf{t} = \mathbf{T}.$$  

(2.2)

The definition of $\Delta$ and a description of its characteristics are given in Appendix B. Briefly, $\Delta$ interchanges the first and last three components of a wrench. This changes a wrench in ray coordinates to one in axis coordinates and vice versa. The operator also has the properties,

$$\Delta^T = \Delta, \quad \Delta^T = \Delta, \quad \Delta \Delta = \mathbf{I}.$$  

Combining (2.1) and (2.2) and making use of the relationship between $\Delta$ and its inverse gives

$$\mathbf{t} = \Delta \mathbf{C} \mathbf{w},$$  

(2.3)

which is an equation in ray coordinates. The corresponding eigenvalue problem is

$$\lambda \mathbf{e} = \Delta \mathbf{C} \mathbf{e},$$  

(2.4)

which yields six eigenvalues $\lambda_i$ and six eigenvectors $\mathbf{e}_i$. The eigenvectors are referred to as eigenscrews and are in ray coordinates. A wrench applied about an eigenscrew yields a twist deformation about the same eigenscrew. Any
deformation permitted by the compliance matrix can be decomposed into deformations about the six eigenscrews.

An equivalent form of the compliance matrix eigenvalue problem can be expressed in axis coordinates using,

\[ \Delta e = E, \quad e = \Delta E. \] (2.5)

Substituting (2.5) into (2.4) gives

\[ \lambda E = C \Delta E. \] (2.6)

This equation is in axis coordinates and is equivalent to (2.4) which is in ray coordinates. It is simple to demonstrate that both (2.4) and (2.6) yield the same eigenvalues and the same eigenscrews.

The relationship between the compliance and the stiffness characteristics of an elastic system is,

\[ S^{-1} = C, \] (2.7)

where \( S \) is the symmetric 6x6 stiffness matrix. This suggests that the stiffness matrix eigenvalue problem is equivalent to the compliance matrix eigenvalue problem. Stated in terms of stiffness, relationship (2.1) becomes

\[ w = S \Delta t. \] (2.8)

Transforming the twist into ray coordinates using (2.2), yields

\[ w = S \Delta f. \] (2.9)

The corresponding eigenvalue problem is

\[ \lambda f = S \Delta f. \] (2.10)

Premultiplying by \((S \Delta)^{-1}\) and using (2.7) gives

\[ \frac{1}{\lambda} f = \Delta C f. \] (2.11)

Comparison of (2.11) with (2.4) shows that \((1/\lambda) = \lambda\) and \(f = e\). Therefore, the compliance and stiffness problems are equivalent and need not be considered separately.

It should be noted that a twist deformation about an eigenscrew of the stiffness matrix produces a wrench on the same eigenscrew. This is the definition Ball gives for the principal screws of the potential. Thus, the stiffness matrix eigenscrews and the principal screws of the potential are identical. Since the compliance matrix eigenscrews and the stiffness matrix eigenscrews are also identical, Ball's principal screws of the potential can be found using either matrix.

The developments in this section show that the compliance matrix eigenvalue problem, in both ray and axis coordinates, and the stiffness matrix eigenvalue problem are identical problems. Therefore, without loss of generality the remaining developments use the compliance matrix and ray coordinates.

3. The Compliance Matrix Eigenstructure

The eigenscrews and the eigenvalues of the compliance matrix possess a number of unique characteristics. In this section the basic relationships between eigenscrews, eigenscrew pitches and eigenvalues are presented. A numerical example illustrates the propositions.

**Proposition 1:** The matrix \( \Delta C \) has three positive and three negative eigenvalues.

**Proof:** The compliance matrix \( C \), is by definition positive definite. It therefore can be expressed as

\[ C = L L^T. \] (3.1)

Substitution into \( \lambda e = \Delta C e \) yields,

\[ \lambda e = \Delta L L^T e. \] (3.2)

Introducing the change of variables \( x = L^T e \) gives,

\[ \lambda x = (L^T \Delta L) x. \] (3.3)

This is a congruence transformation of \( \Delta \). By Sylvester's Law of Inertia, the number of positive, negative, and zero eigenvalues is invariant under non-singular congruence transformations. Since the operator \( \Delta \) has three positive and three negative eigenvalues, the same is true for \( \Delta C \).

**Proposition 2:** The matrix \( \Delta C \) has a full set of eigenscrews.

**Proof:** Returning to (3.3), \( L^T \Delta L \) is a symmetric matrix. It therefore has a full set of eigenvalues satisfying

\[ \lambda_i f_i = L^T \Delta L f_i, \quad \lambda_i f_i = 0. \] (3.4)

The change of variables relation, \( e_i = L^T x_i \) demonstrates the one-to-one correspondence between the eigenvectors. Since, \( L^T \Delta L \) has a full set of eigenvectors, \( \Delta C \) also has a full set of eigenscrews.

**Proposition 3:** All eigenscrews of \( \Delta C \) are non-isotropic, i.e. all the eigenscrews have non-zero, finite pitches,

\[ e_i^T \Delta e_i \neq 0. \] (3.5)

**Proof:** The compliance eigenvalue problem (2.4) is multiplied by \( e^T \Delta \) to give

\[ \lambda e^T \Delta e = e^T C e. \] (3.6)

Since \( C \) is positive definite, then by definition \( e^T C e > 0 \). Proposition 1 states that \( \lambda \) is non-zero and finite. Thus, \( e^T \Delta e \) is also non-zero and finite.

**Proposition 4:** The eigenscrews of \( \Delta C \) are coreciprocal.

\[ e_i^T \Delta e_j = 0, \quad i \neq j, \quad i, j = 1, \ldots, 6. \] (3.7)

**Proof:** The eigenvalue problem is first written for two distinct eigenvalues, \( \lambda_i \neq \lambda_j \),

\[ \lambda_i - \Delta C e_i = 0 \] (3.8)

\[ \lambda_j - \Delta C e_j = 0. \] (3.9)

Forming \( e_j^T \Delta (3.8) - e_i^T \Delta (3.9) \) gives

\[ (\lambda_i - \lambda_j) e_i^T \Delta e_j = 0. \] (3.10)

Since \( \lambda_i \neq \lambda_j \), then \( e_i^T \Delta e_j = 0 \), and the eigenscrews are coreciprocal. If the eigenvalues are equal, the corresponding eigenscrews \( (e_i^T, e_j^T) \), \( k = 1, \ldots, m \), may be made coreciprocal via a process analogous to a Gram-Schmitt orthogonalization, Sugimoto and Duffy (1982).

**Proposition 5:** For each eigenscrew of the matrix \( \Delta C \), the corresponding eigenvalues and eigenscrew pitches have the same sign.

**Proof:** Assume that each eigenscrew is normalized as a unit screw, then

\[ e_i^T L e_i = e_i^T \begin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix} e_i = 1. \] (3.11)

The pitch is given by
\[ h_i = \frac{1}{2} e_i^T \Delta e_i. \]  
(3.12)

Rewriting (3.6) with \( e_i \) and using (3.11) yields
\[ h_i \lambda_i = \frac{1}{2} e_i^T C e_i. \]  
(3.13)

Since the matrix \( C \) is positive definite \( e_i^T C e_i > 0 \) and \( h_i \lambda_i > 0 \). This requires that \( h_i \) and \( \lambda_i \) have the same sign. Proposition 1 states that the matrix \( \Delta C \) has three positive and three negative eigenvalues. As a result, \( \Delta C \) always has three eigenscrews with positive pitch and three eigenscrews with negative pitch.

A general numerical example illustrates Propositions 1 through 5. The elements of the compliance matrix are provided by a random number generator. The matrix was verified to be positive definite.

**Example 3.1** Verification of Propositions 1 through 5.
A general compliance matrix with all random elements is,
\[
C = \begin{bmatrix}
8.50 & 0.50 & -1.30 & -2.70 & 0.50 & 0.40 \\
-0.50 & 10.40 & -5.30 & -1.00 & -0.10 & 2.90 \\
-1.30 & 0.50 & 10.00 & -0.80 & -0.30 & -0.50 \\
-2.70 & 1.00 & -0.80 & 10.60 & 2.40 & 2.30 \\
0.50 & -0.10 & -0.30 & 2.40 & 6.70 & 2.70 \\
2.70 & 2.90 & -0.50 & 2.30 & 2.30 & 4.70 \\
\end{bmatrix}.
\]

Solving the eigenvalue problem (2.4) yields the eigenscrews \( [e] \), the eigenvalues \( [\lambda] \), and the pitches \( [h] \), where each is a 6x6 matrix. The diagonal form of the pitch matrix indicates that the six eigenscrews are coregionalist.

\[ [e] = \begin{bmatrix}
-0.664 & 0.417 & 0.014 & -0.355 & -0.286 & 0.444 \\
-0.264 & 0.481 & 0.411 & -0.138 & 0.360 & -0.419 \\
0.057 & 0.281 & 0.228 & 0.313 & -0.495 & 0.325 \\
0.564 & 0.314 & 0.276 & -0.466 & 0.257 & 0.335 \\
0.317 & 0.566 & -0.471 & 0.238 & -0.503 & -0.479 \\
-0.142 & 0.249 & -0.693 & 0.696 & 0.474 & 0.425 \\
\end{bmatrix}. \]


\[ [h] = \frac{1}{2} e_i^T \Delta e_i, \]

\[ = \text{diag} [-0.909, 1.007, -1.575, 1.440, -1.072, 1.020]. \]

Summarizing, several fundamental properties have been demonstrated in this section. In the previous relevant work, Ball (1900) shows that the compliance matrix eigenscrews are coregionalist. He also shows that the eigenscrews and eigenvalues are real.

**4. Generalized Compliant Axes**

In practice there are few direct applications for the eigenscrews of the compliance matrix. A wrench applied about an eigenscrew produces a twist on the same eigenscrew. However, the practical applications of this to an elastic system are not clear. In special cases, the eigenscrews may define higher order invariant subspaces which can be used for compliance matrix classification. Since there are always six eigenscrews, the necessary and sufficient condition for the existence of such subspaces is the occurrence of repeated eigenvalues. As with the eigenscrews themselves, the relationship between the subspaces and the characteristics of a manipulator is not readily apparent.

There is a need for an alternative description of the compliance matrix properties, one which provides information of a more practical nature. One such a description is the compliant axis concept. A rotational deformation applied about a compliant axis produces a couple in the direction of the axis and a force applied along the same compliant axis produces a linear deformation in the direction of the axis. The compliant axis acts as a torsional spring for rotational deformations applied about the axis and as a linear spring for forces applied along the axis. For example, a manipulator which has a compliant axis perpendicular to a work surface is ideally suited for gripping operations. Reaction forces perpendicular to the surface cause displacements in the same direction, thus making control of the surface finish a much simpler task. A limitation of the compliant axis concept is that not all compliance matrices have them. A detailed discussion of compliant axes and their use in the classification of compliance matrices is given by Patterson and Lipkin (1990).

A first generalization of the compliant axis concept is that of a partial compliant axis. Essentially it is one-half of a compliant axis. There are two types of partial compliant axes, a rotational-compliant axis and a force-compliant axis. A rotational deformation applied about a rotational-compliant axis produces a couple in the direction of the axis. Analogously, a force applied along a force-compliant axis produces a linear deformation in the direction of the axis. Defining the conditions for the existence of these partial compliant axes is a difficult problem which is currently under investigation. However, as shown in the following examples, the force- and rotational-compliant axes do exist.

**Example 4.1:** Rotation-compliant Axis
The compliance matrix is
\[
C = \begin{bmatrix}
0.86 & 0.21 & 0 & 0 & 0 & 0 \\
0 & 1.98 & 0.37 & 0 & 0 & 0 \\
0.21 & 0.37 & 0.99 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.37 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.08 & 0 \\
0 & 0 & 0 & 0 & 0 & 2.06 \\
\end{bmatrix}.
\]

Using \( t = \Delta C w \), (2.3) and applying a force and then a torque in the x-direction yields,
\[
\begin{bmatrix}
0.88 \\
0.98 \\
1.37 \\
0.24 \\
\end{bmatrix} = \Delta C \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}.
\]

Thus the x-axis is a rotation-compliant axis, but not a force-compliant axis. It is not a force-compliant axis because the resulting linear deformation is not parallel to the applied force.

**Example 4.2:** Force-compliant Axis
The compliance matrix is
\[
C = \begin{bmatrix}
1.65 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.83 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.44 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.99 & 0.30 & 1.05 \\
0 & 0 & 0 & 0.30 & 1.17 & 1.11 \\
0 & 0 & 0 & 1.05 & 1.11 & 2.04 \\
\end{bmatrix}.
\]

Using (2.3) and applying a force and then a torque in the x-direction yields,
Thus the x-axis is a force-compliant axis, but not a rotation-compliant axis. It is not a rotation-compliant axis because the resulting rotation is not parallel to the applied couple.

The concept of rotation- and force-compliant axes admits a further generalization. This results in the twist-compliant and the wrench-compliant axes. A twist deformation applied about a twist-compliant axis produces a pure couple in the direction of the axis. A wrench applied about a wrench-compliant axis produces a pure linear deformation in the direction of the axis. The twist- and wrench-compliant axes are each determined by solving an eigenvalue problem. The remainder of this section is devoted to the derivation of these two new eigenvalue problems. The results are then used to present several propositions characterizing the twist- and wrench-compliant axes.

The twist-compliance axes are found using the stiffness relationship \( w = C^{-1} \Delta t \), (2.9) where \( C^{-1} = S \). A twist deformation and a pure couple in the direction of the twist are related by,

\[
\eta_b \begin{bmatrix} 0 \\ \xi \end{bmatrix} = C^{-1} \Delta \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \tag{4.1}
\]

where \( t = [\xi^T \eta_{b1}^T]^T \), \( w = \eta_{b1} [0^T \xi^T]^T \), \( \xi \) is the twist direction, and \( \eta_{b1} \) is the couple magnitude. Reformulation of (4.1) as an eigenvalue problem, enables the determination of the twist deformation,

\[
\eta_b \Delta \Gamma t = C^{-1} \Delta t \tag{4.2}
\]

where

\[
\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

A brief description of the operator \( \Gamma \) is provided in Appendix B. Rearranging (4.2) gives,

\[
\begin{bmatrix} 0 \\ \xi \end{bmatrix} = (\Delta C \Delta \Gamma \eta_{b1}^{-1}) t. \tag{4.3}
\]

Equation (4.3) is an eigenvalue problem with six eigenvalues \( \eta_{b1} \) (1/\( \eta_{b1} \)) and six eigenscrews \( \eta_{b1} t \). The subscript \( t \) is used to distinguish the eigenscrews from eigenscrews of this problem from those of the previous compliance matrix problem.

It is easy to demonstrate that \( \Delta C \Delta \Gamma \) is rank three and thus has three zero eigenvalues. In this case (4.3) reduces to \( \Gamma t = 0 \) and the corresponding eigenscrews must be pure linear deformations. By (4.1) \( \eta_{b1} \) is the couple magnitude and thus a zero eigenvalue (1/\( \eta_{b1} \)) results in a couple \( w \) of infinite magnitude. From \( w = C^{-1} \Delta t \) it follows that the linear deformation \( t \) also has an infinite magnitude. Since an infinite magnitude is required, these three eigenscrews are not considered further.

The three remaining finite eigenscrews correspond to the non-zero eigenvalues. These eigenscrews are the twist deformations which produce pure couple in the direction of the eigenscrew axes. For these couples the stiffness relationship becomes,

\[
w_{b1} = C^{-1} \Delta t_{b1} \quad i = 1, 2, 3 \tag{4.4}
\]

where

\[
w_{b1} = \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \quad t_{b1} = \begin{bmatrix} \eta_{b1} \\ \eta_{b1} \xi_{b1} \end{bmatrix}.
\]

The axes of the eigenscrews are the twist-compliant axes.

Analogously, the wrench-compliant axes are determined using the compliance relationship \( t = \Delta w \), (2.3). A wrench and a pure linear deformation in the direction of the wrench are related by,

\[
\eta_c \begin{bmatrix} 0 \\ y \end{bmatrix} = C \begin{bmatrix} x \\ y_0 \end{bmatrix}, \tag{4.5}
\]

where \( w = [y^T y_0^T]^T \), \( t = \eta_c [0^T y^T]^T \), \( y \) is the direction of the wrench, and \( \eta_c \) is the magnitude of the linear deformation. Reformulation of (4.5) as an eigenvalue problem enables the determination of the wrench,

\[
\eta_c \Delta \Gamma w = \Delta C w \tag{4.6}
\]

Rearranging and simplifying (4.6) gives

\[
\begin{bmatrix} 0 \\ y \end{bmatrix} = (\Delta C \Delta \Gamma \eta_c^{-1}) w. \tag{4.7}
\]

Equation (4.7) is an eigenvalue problem with six eigenvalues \( \eta_{c1} \) (1/\( \eta_{c1} \)) and six eigenscrews \( \eta_{c1} w \). The subscript \( w \) is used to distinguish the eigenvalues and eigenscrews of the wrench-compliant axis problem from those of the two previous eigenvalue problems.

It is easy to demonstrate that \( \Delta C \Delta \Gamma \) is rank three, and thus has three zero eigenvalues. In this case (4.7) reduces to \( \Gamma w = 0 \) and the corresponding eigenscrews are pure couples. By (4.5) \( \eta_{c1} \) is the linear deformation magnitude and thus a zero eigenvalue (1/\( \eta_{c1} \)) results in a linear deformation \( t \) of infinite magnitude. From \( t = \Delta w \) it follows that the couple \( w \) also has infinite magnitude. Since an infinite magnitude is required, these three eigenscrews are not considered further.

The remaining finite eigenscrews correspond to the non-zero eigenvalues. These eigenscrews are the wrenches which produce linear deformations in the direction of the eigenscrew axes. For these linear deformations the compliance relationship becomes

\[
t_{b1} = \Delta C w_{b1} \quad i = 1, 2, 3 \tag{4.8}
\]

where

\[
t_{b1} = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}, \quad w_{b1} = \begin{bmatrix} 1 \\ \eta_{c1} y_{b1} \end{bmatrix}.
\]

The axes of the eigenscrews are the wrench-compliant axes.

The wrenches and twist deformation of (4.4) and (4.8) possess a number of interesting properties which are detailed in the following propositions. The first three propositions concern the twist-compliant axes, and the second three the wrench-compliant axes. The last two concern the reciprocity relationships between the wrenches and twists of these compliant axes.
Proposition 6: The couples $w_{ti}$ and the axes of the twist deformations $t_{ti}$ are in the same direction as the eigenvectors of the $3 \times 3$ matrix $C$, where
\[
C = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}.
\] (4.9)

Proof: From (4.4) a unit couple is expressed as
\[
w_{ti} = \begin{bmatrix} 0 \\ x_i \end{bmatrix},
\] (4.10)
and multiplying with $\Delta C$ gives,
\[
t_{ti} = \Delta C w_{ti} = \begin{bmatrix} c x_i \\ b x_i \end{bmatrix}.
\] (4.11)
Since $t_{ti}$ must be in the direction of $x_i$, then $c x_i = \mu_i x_i$ where $x_i$ and $\mu_i$ are the eigenvector directions and eigenvalues of the matrix $C$.

Proposition 7: The couples $w_{ti}$ and the axes of the twists $t_{ti}$ are each orthogonal.

Proof: The matrix $c$ in (4.9) is by definition symmetric and of full rank, therefore it has three orthogonal eigenvectors. By Proposition 6, the axes of the couples $w_{ti}$ and the axes of the twists $t_{ti}$ are in the same direction as these eigenvectors.

Proposition 8: The eigenvalues $\eta_i (1/\eta_i)$ corresponding to $t_{ti}$ are positive and equal to the eigenvalues $\mu_i$ of the matrix $C$ in (4.9).

Proof: Substituting the eigenvector expression $c x_i = \mu_i x_i$ into (4.11) gives,
\[
\begin{bmatrix} \mu_i x_i \\ b x_i \end{bmatrix} = \Delta C w_{ti}.
\] (4.12)
Rearranging (4.4) gives,
\[
\frac{1}{\eta_i} \begin{bmatrix} x_i \\ x_0 \end{bmatrix} = \Delta C w_{ti}.
\] (4.13)
Comparison of (4.12) and (4.13) shows that $(1/\eta_i) = \mu_i$. Since $C$ is positive definite the $(1/\eta_i)$ is positive.

The next three propositions concern the wrench-compliant axes and have been previously addressed by Dimentberg (1965). These propositions and corresponding proofs are analogous to those for the twist-compliant axes. For the sake of brevity the proofs are omitted.

Proposition 9: The linear deformations $t_{bi}$ and the axes of the wrenches $w_{bi}$ are in the same direction as the eigenvectors of the $3 \times 3$ matrix $d$, where
\[
C^{-1} = S = \begin{bmatrix} d & g \\ g^T & f \end{bmatrix}.
\] (4.14)

Proposition 10: The linear deformations $t_{bi}$ and the axes of the wrenches $w_{bi}$ are each orthogonal.

Proposition 11: The eigenvalues $\eta_i (1/\eta_i)$ corresponding to $w_{bi}$ are positive and equal to the eigenvalues $\mu_i$ of the matrix $d$ in (4.14).

The remaining two propositions concern the reciprocity relationships between the twists and wrenches of the twist- and wrench-compliant axes.

Proposition 12: The twists $t_{ti}$ of the twist-compliant axes are coreciprocal to the wrenches $w_{bi}$ of the wrench-compliant axes, $t_{ti}^T \Delta w_{bi} = 0$, $i, j = 1, 2, 3$.

Proof: The eigenvalue equations (4.3) and (4.7) are rewritten as
\[
\eta_i \Delta C \Delta \Gamma t_{ti} = \eta_i t_{ti},
\] (4.15)
\[
\eta_i \Delta C \Delta \Gamma w_{bi} = \eta_i w_{bi}.
\] (4.16)
Forming the equation $t_{ti}^T \Delta w_{bi}$ and simplifying gives,
\[
t_{ti}^T \Delta w_{bi} = (\eta_i \eta_j) (t_{ti}^T (\Delta \Gamma) w_{bi}).
\] (4.17)
Identically $\Gamma^T \Delta \Gamma = 0$, yielding $t_{ti}^T \Delta w_{bi} = 0$.

Proposition 13: The couples $w_{ti}$ of the twist-compliant axes and the linear deformations $t_{bi}$ of the wrench-compliant axes are coreciprocal.

Proof: The couples $w_{ti}$ are free vectors and span a three-system. Similarly, the linear deformations $t_{bi}$ are also free vectors and span a three-system, Hunt (1978). All free vectors belong to the same special three-system which is self-reciprocal (i.e. isotropic).

The relationship between the twist deformations $t_{bi}$, $t_{ti}$ and the wrenches $w_{bi}$, $w_{ti}$ is shown in Figure 2. In Figure 2, the bold lines connect corresponding twist deformations and wrenches. The dashed lines connect coreciprocal quantities. Ball (1900) defines a screw pair known as the conjugate screws of the potential. Suppose a twist $\theta$ produces a wrench $\eta$ and a twist $\varphi$ produces a wrench $\xi$. If $\theta$ is reciprocal to $\xi$, then $\varphi$ is reciprocal to $\eta$. The twists $\theta, \varphi$ are on conjugate screws of the potential. By this

![Figure 2. Relationship Between Twist Deformations and Wrenches](image)

definition $t_{bi}$ and $t_{ti}$ form conjugate screw systems of the potential. Throughout Ball's work, the concept of conjugate screws are used to characterize the properties of screw systems.

The work of Dimentberg (1965) is closely related. He discusses what are referred to here as wrench-compliant axes, but does not mention twist-compliant axes. Dimentberg determines the directions of the three orthogonal wrench-compliant axes by solving the eigenvalue problem for the $3 \times 3$ upper diagonal block of the
stiffness matrix, \( d \) in (4.14). The wrenches on these axes are determined by multiplying a unit linear deformation, in the direction of each axis, with the stiffness matrix. An eigenscrew formulation, such as (4.7) is not used.

Dimentberg does show that when the compliant wrench axes are taken in pairs and a pure rotation is applied about the common perpendiculars, then a set of three orthogonal wrenches is produced. The axes of the wrenches produced by the rotations are parallel to the wrench-compliant axes. The linear deformations along the wrench-compliant axes and the rotations about the common perpendiculars do not form conjugate screw systems. The common perpendicular about which a rotation is applied, intersects only two wrench-compliant axes. The rotation can only be shown to be reciprocal with the wrenches on these two axes. It is not necessarily reciprocal to the wrench on the third compliant axis.

Summarizing, compliant axes fall into three categories, from least to most general these are:

1. **Compliant Axis** - A rotational deformation produces a couple in the direction of the rotation and a force along the axis of the rotation produces a linear deformation in the same direction.

2. a. **Rotation-compliant Axis** - A rotational deformation produces a couple in the direction of the rotation.
   b. **Force-compliant Axis** - A force produces a linear deformation in the direction of the force.

3. a. **Twist-compliant Axis** - A twist produces a couple in the direction of the twist.
   b. **Wrench-compliant Axis** - A wrench produces a linear deformation in the direction of the wrench.

**Concluding Remarks**

The principles of screw theory are applied to the compliance matrix eigenvalue problem for an unconstrained, elastically suspended rigid body. This approach makes it relatively simple to show the equivalence of the compliance and stiffness matrix eigenvalue problems. The eigenstructure of the compliance problem yields basic propositions which characterize the relationship between the eigenscrews, the eigenscrew pitches and the eigenvalues. A new type of eigenvalue problem is formulated and is used to determine the twist- and wrench-compliant axes. These axes are shown to specify conjugate screw systems of the potential. Twist- and wrench-compliant axes are generalizations of the compliant axis concept. These concepts lend themselves to practical robot operations as they define the directions and orientations of “simple” end effector reactions resulting from contact with the environment.

This paper makes two major contributions. First, through the application of screw theory a number of new propositions concerning the properties of the compliance matrix eigenstructure are proven. Second, a new type of eigenvalue problem is formulated which generalizes Dimentberg’s wrench-compliant-axes to include twist-compliant axes. Together they form conjugate screw systems of the potential.

**Appendix A** Given a rigid body suspended by \( n \) linear springs, the stiffness relationship \( w = St \), (2.8) describes the elastic characteristics of the system. The wrenches produced by applying a twist to the body are given by

\[
\mathbf{w} = \sum \mathbf{w}_i = \sum \mathbf{u}_i \mathbf{a}_i = \mathbf{u} \mathbf{a}.
\]

where \( \mathbf{u}_i = [\mathbf{u}_i^T \mathbf{u}_i v_i^T]^T \) are the unit screws in the direction of the \( n \) linear springs, and \( \mathbf{a}_i \) are the corresponding force magnitudes. Placing the origin on the \( i^{th} \) linear spring axis results in, \( \mathbf{u}_i = [\mathbf{u}_i^T 0]^T \). The deformation is given by

\[
\mathbf{t} = [\mathbf{q}_i^T \mathbf{q}_j^T]D. 
\]

Therefore the linear deformation in the direction of \( \mathbf{u}_i \) is

\[
\mathbf{q} \cdot \mathbf{u}_i = \mathbf{u}_i^T \mathbf{D} \mathbf{t}.
\]

The term \( \mathbf{u}_i^T \mathbf{D} \mathbf{t} \) is invariant with respect to coordinate frame transformations. The force along each spring axis, with spring rate \( k_i \), is thus

\[
k_i (\mathbf{q} \cdot \mathbf{u}_i) = k_i \mathbf{u}_i^T \mathbf{D} \mathbf{t} = \mathbf{a}_i.
\]

In matrix form \( \mathbf{a}_i \) becomes,

\[
\mathbf{a} = \mathbf{K} \mathbf{u}^T \mathbf{D} \mathbf{t},
\]

where \( \mathbf{K} \) is a diagonal matrix of spring rates. Substituting \( \mathbf{a} \) into \( \mathbf{w} = \mathbf{u} \mathbf{a} \) yields

\[
\mathbf{w} = \mathbf{u} \mathbf{K} \mathbf{u}^T \mathbf{D} \mathbf{t},
\]

which simplifies to

\[
\mathbf{w} = \mathbf{S} \mathbf{D} \mathbf{t},
\]

where

\[
\mathbf{S} = \mathbf{u} \mathbf{K} \mathbf{u}^T.
\]

The formulation is the same if torsional springs replace the linear springs.

**Appendix B** The matrix, \( \mathbf{D} \) can be partitioned into \( 3 \times 3 \) matrices as,

\[
\mathbf{D} = \begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix}
\]

It is an orthogonal involution,

\[
\mathbf{D} = \mathbf{D}^T, \quad \mathbf{D} = \mathbf{D}^{-1}, \quad \mathbf{D} \mathbf{D} = \mathbf{I}.
\]

The \( \mathbf{D} \) operator transforms a screw from ray to axis coordinates and vice versa. The \( \mathbf{D} \) operator has three positive and three negative eigenvalues, as shown by the following decomposition,

\[
\mathbf{D} = \mathbf{Q} \mathbf{T} \mathbf{D} \mathbf{Q}^{-1}
\]

where

\[
\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]

Since \( \mathbf{Q}^T \mathbf{Q} = \mathbf{I} \), then the columns of \( \mathbf{Q} = \mathbf{Q}^T \) are the eigenvectors of \( \mathbf{D} \). The three positive and three negative elements of \( \mathbf{D} \) are the eigenvalues of \( \mathbf{D} \).

The matrix \( \mathbf{D} \) is given by

\[
\mathbf{D} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]

321
The operator $\Gamma$ converts a screw in ray coordinates to a free vector in axis coordinates. It can be used to find the magnitude of a screw in ray coordinates,

$$mag(\mathbf{x}) = (\mathbf{x}^T \Gamma \mathbf{x})^{1/2}.$$  

Lipkin and Duffy (1988) gives a discussion of $\Gamma$ as the Euclidean projective metric.

Bibliography


Kazerooni, H., 1988, "Direct-Drive Active Compliant End Effector (Active RCC)," *Proceedings IEEE International Conference on Robotics and Automation*.


CAM S, GEARS, ROBOT AND MECHANISM DESIGN

presented at
THE 1990 ASME DESIGN TECHNICAL CONFERENCES — 21st BIENNIAL MECHANISMS CONFERENCE
CHICAGO, ILLINOIS
SEPTEMBER 16–19, 1990

sponsored by
THE MECHANISMS COMMITTEE OF THE DESIGN ENGINEERING DIVISION, ASME

edited by
A. PISANO
UNIVERSITY OF CALIFORNIA–BERKELEY

M. McCARTHY
UNIVERSITY OF CALIFORNIA–IRVINE

S. DERBY
RENSSELAER POLYTECHNIC INSTITUTE

THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 East 47th Street □ United Engineering Center □ New York, N.Y. 10017