CHAPTER 6
SUMMARY AND CONCLUDING REMARKS

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery.  R. Descartes [1637, p. 240]

A systematic presentation of geometric principles and methods has been developed from basic concepts. In Chapter 2, projective geometry was used as the starting point of the development. In order to work analytically from the outset, projective coordinates were introduced in a metric-free manner. This was then followed by a more general development of projective homogeneous coordinates using symmetrical determinant principles for points, planes and lines which were then extended to screws. This enabled a rather elegant description of dualistic properties. Projective transformations were introduced as those which left incidence properties invariant. The transformations for points, planes, lines and screws were united using an important tetrahedron principle based on incidence relations.

The systematic development continues in Chapter 3 for metrical geometry which is based directly on Cayley's principle of the Absolute. This allows a smooth and logical transition from projective geometry to metrical geometry via projective metrics. Following suggestions made by Clifford, determination of the pitches and axes of screws was generalized. This was a rediscovery of an earlier result given by Buchheim [1884a] who apparently was also directed by Clifford's work. Metrical collineations were introduced based upon the invariance of the Absolute. Further, definitions of norms and metrical coordinates enabled the introduction of new space elements with a magnitude. Elliptic geometry was detailed with the main result being to introduce the elliptic polarity. Projective and elliptic relations were identified for screws which appear similar in form. They are easily confused without thorough geometric understanding and an accompanying notation that delineates correlations from collineations and ray coordinates from axis coordinates.

Euclidean geometry was shown to have an asymmetry in its representation of duality because of the singular nature of the Euclidean Absolute. An elegant formulation for the pitch and axis of a screw was derived from Clifford's general principle. Most importantly, it appears that a new contribution has been made in the well-known subject of Euclidean geometry. That is, by a direct application of Cayley's Absolute, the group of Euclidean collineations is immediately deduced, namely translations, rotations and reflections through the origin.

Vector quantities were deduced based on one of Klein's principles. This allowed the introduction of new space elements which have both an associated magnitude and direction. In a manner, they enabled a resolution of the ambiguity
of signs associated with metrical coordinates. The vector formulation of screws led to a form which is presently common. The role of twists and wrenches in mechanics was discussed along with the principle of virtual work.

Chapter 4 began with summary of the circumstances which motivated the work in this dissertation. Virtually all the material in this chapter is entirely original unless specifically noted otherwise. Important contributions include:

1. A basic mapping of screws onto the quadruple \((h, \rho)\).

2. A quaternion mapping where the elliptic polarity induces a quaternion inverse. The operator role of the quaternion in relation to a screw and its polar is deduced.

3. A mapping of screws onto a four-dimensional inverse space where the elliptic polarity induces an inversion through the unit hypersphere followed by a reflection in a hyperplane.

4. A mapping of the quadruple \((h, \rho)\) onto radial pencils and bundles of screws with constant pitch.

5. A three-way isomorphism between quaternions, points in an inverse space and radial pencils and bundles of screws.

6. A relation between pairs of polar helices and their torsion and curvature vectors.

Further, many of these developments were applied to Ball’s planar representation of the two-system and contributions in this area include:

1. The representation was generalized to the inversive plane where the elliptic polarity induced an inversion through the unit circle followed by a reflection.

2. Points on the plane were characterized as radial pencils of screws and circles were characterized as pencils of cyldroids.

3. Specializations of quaternions represented ordinary complex numbers on the inversive plane.

4. A general derivation of the circle representation for the two-system was presented.

5. An examination of the five special two-systems in terms of their mappings.

6. Self-polar two-systems were deduced in terms of the planar mapping.

7. The apparently erroneous results from a complex number formulation were explained using the interpretation of the generalized planar mapping and exemplified its utility.

8. A special planar two-system was used to explain the effect of an origin translation on the elliptic polar of a screw.

Chapter 5 began with an explanation of hybrid control using the concept of kinestatic filtering. This was followed by an examination of a type of noninvariant hybrid control and various types of invariant kinestatic filtering for application in hybrid control. Important contributions in this area include:

1. A demonstration that "orthogonal" projection is noninvariant with respect to Euclidean translations.

2. A demonstration that "orthogonal" projection is noninvariant with changes in unit length.

3. The determination of origin locations as a quadric surface where two screws are "orthogonal."

4. Three proposed necessary conditions for invariant kinestatic filtering.
5. The introduction of a family of invariant kinesthetic filters using a variable parameter.

6. A detailed discussion on the applicability of the introduced invariant filters.

Further, the methods used for determining invariant kinesthetic filters indicate that many more solutions may be obtained by optimizing invariant criteria.

It is the author's hope that the geometrical developments presented will provide a foundation for further utilization and popularization of screw theory in kinematics. Further, it is believed that the elucidation of the properties of the elliptic polarity and "orthogonal" projection will have a significant impact on the present noninvariant theory of hybrid control for robotic manipulators. Clearly, invariant methods of hybrid control are needed and this dissertation provides the geometrical foundations for their development as well as introducing initial invariant formulations which may be extended by optimization techniques.