CHAPTER 2
PROJECTIVE GEOMETRY

Throughout the ages, from the ancient Egyptians and Euclid to Poncelet and Steiner, geometry has been based on the concept of measurement, which is defined in terms of the relation of congruence. It was von Staudt (1798–1867) who first saw the possibility of constructing a logical geometry without this concept. Since his time there has been an increasing tendency to focus attention on the much simpler relation of incidence, which is expressed by such phrases as "The point A lies on the line p" or "The line p passes through the point A." H.S.M. Coxeter [1942, p. 18]

During the nineteenth century a great deal of investigation centered about geometrical relations which are independent of metrical concepts such as distance, angle, area, etc. Metric-free geometric relations form the subject of projective geometry of which the property of incidence is paramount, such as a point is incident to a plane or a line cuts a conic. For many years projective geometry was viewed as a relatively insignificant area within the domain of Euclidean geometry. The situation took a radical turn in 1859 when Cayley demonstrated that projective geometry was actually the most general and that Euclidean geometry was merely a specialization where a certain configuration was deemed fixed and used as a reference for measurement. Later, Klein demonstrated how non-Euclidean geometries could be included under Cayley's principle which is explained in Chapter 3.
Thus, there arose a contradiction. Since coordinates were typically based upon metrical considerations, how could such coordinates be logically applied to projective relations since metrical geometry is subordinate? Klein supplied an answer to this by suggesting the use of von Staudt's projective constructions which are employed to define the algebra of points and is presented in a modified form in Section 2.1. First nonhomogeneous coordinates are introduced which are then used to define the homogeneous coordinates of points.

In Section 2.2 determinant formulations are utilized to develop homogeneous coordinates in the plane for points and lines and in space for points, planes, lines and screws following extensional principles attributable to Grassmann. This enables a rather elegant and complete symmetry to be displayed amongst dual elements. Plucker's ray and axis line coordinates are detailed and their important identical relation is formulated which is subsequently used throughout the entire work.

Projective transformations of space preserve incidence relations and are detailed in Section 2.3. An important tetrahedron relationship is introduced which unifies projective transformations for points, planes and lines or screws. Polarities are developed as correlations whose properties are symmetrical with their inverses and are then employed in Chapter 3.

Section 2.1 A Projective Formulation of Coordinates

In the first place, it is important to realize that when coordinates are used, in projective Geometry, they are not coordinates in the ordinary metrical sense, i.e. the numerical measures of certain spatial magnitudes. On the contrary, they are a set of numbers, arbitrarily but systematically assigned to different points, like the numbers of houses in a street, and serving only, from a philosophical standpoint, as convenient designations for points which the investigation wishes to distinguish. But for the brevity of the alphabet, in fact, they might, as in Euclid, be replaced by letters. Bertrand Russell [1897, pp. 118-119]

Projective geometry deals with the incidence properties of elements and figures without employing any form of measurement or metric. However, various metrical subgeometries may be derived from the more general projective geometry by requiring that certain figures be designated as an absolute reference. Thus to employ metrical concepts, such as Cartesian coordinates, in the analytical development of projective geometry is to incur a logical contradiction since metrical relations are subordinate.

In this section, it is demonstrated that coordinates may be introduced into projective geometry in a metric-free manner. Coordinates are important in the analytical development of projective geometry since they are the tools for the derivation of compact expressions for complex relations and further, they provide insight which may not be evident from a synthetic development. Coordinates also provide an essential vehicle for numerical calculations which may be used to verify and illustrate relations unambiguously. Since coordinates are routinely
used in metrical geometries it is highly desirable to demonstrate how they may be first utilized projectively and then specialized for metrical application.

The introduction of coordinates in projective geometry presented here is not intended to be an exhaustive treatment but rather an illustrative one. The intent is to draw a sharp distinction between projective and metrical concepts at the outset and to provide a logical sequence of development from projective to metrical geometry. The analysis in this section is based primarily on the complete and systematic methods of Veblen and Young [1910, 1917] to whom frequent reference is made.

It was however von Staudt [1857] who first demonstrated that analytic methods may be introduced into geometry on a strictly projective basis. For this purpose he invented the algebra of throws. The method was later simplified by Hessenberg [1905] and is presented here. First, the addition and multiplication of points on a line is defined using two special projective constructions. Then this algebra of points is demonstrated to be isomorphic to the field of real numbers and is extended to include the concept of infinity in a consistent manner. A unique real number is associated with each point on the line with the exception of a single point which assumes a correspondence with infinity. The unique real number associated with each point is called the nonhomogeneous coordinate of the point on the line. The apparently exceptional role of the point associated with infinity is removed upon the introduction of homogeneous coordinates.

In order to assign coordinates to points on a line \( U \), it is first required to select three distinct points \( x_0, x_1 \) and \( x_\infty \) which together are referred to as a scale. By the special nature of the constructions which define addition and multiplication, the points of the scale are endowed with the properties of 0, 1 and \( \infty \). Figures 2.1.1 and 2.1.2 illustrate the constructions for addition and multiplication which can also be used respectively to define subtraction and division.

Using the operation of addition it is possible to label all the points corresponding to integers, e.g. \( x_1 + x_1 = x_2 \), \( x_1 + x_2 = x_3 \), etc. Next, using the integers with the operation of division it is possible to label all points corresponding to rational numbers. Thus it can be shown that the points corresponding to rational numbers together with the two constructions are isomorphic to the field of rational numbers which includes the properties of associativity, commutivity, distributivity and the existence of inverses.

Next, it is necessary to show that for all points on the line there is a corresponding real number and that together with the constructions they are isomorphic to the field of real numbers. This transition is made from the rational points by invoking a Dedekind cut which is detailed in Coxeter [1942], Veblen and Young [1910].

For the notation used here, the nonhomogeneous coordinates of the points \( x_0, x_1, x_3, \ldots \) are given by the subscripts 0, 1, a, \ldots where the point \( x_\infty \) is excluded. It is important to note that the constructions yield
Figure 2.1.1 Addition of two points. A fixed line $U_0$ through $X_0$ meets the two distinct fixed lines $U_\infty$ and $\overline{U}_\infty$ in points $r$ and $s'$ respectively. The lines $x_a r$ and $x_b s'$ meet $U_\infty$ and $\overline{U}_\infty$ respectively at $r'$ and $s$. The line $r'$ meets $U$ at $x_{a+b}$ which yields the sum of the two points, $x_a + x_b = x_{a+b}$. By reversing the latter steps, subtraction is analogously constructed, eg. $x_a = x_{a+b} - x_b$.

Figure 2.1.2 Multiplication of two points. Through the points $X_0$, $X_1$, $X_\infty$ are drawn respectively the fixed lines $U_0$, $U_1$, $\overline{U}_\infty$ with $U_0$, $U_1$ meeting at $r$ and $U_1$, $\overline{U}_\infty$ meeting at $s$. The lines $x_a r$ and $x_b s'$ meet $U_\infty$ and $U_0$ respectively at $r'$ and $s$. Line $r'$ meets $U$ at $x_{ab}$ which yields the product of the two points, $x_a \cdot x_b = x_{ab}$. By reversing the latter steps, division is analogously defined, eg. $x_a = x_{ab} \div x_b$. 
\[ x_a + x_\infty = x_\infty \quad (1) \]
\[ x_a - x_\infty = x_\infty \quad (x_a \neq x_\infty) \quad (2) \]
\[ x_a - x_\infty = x_\infty \quad (x_a \neq x_0) \quad (3) \]

which are consistent with the usual properties associated with infinity.

Although the three distinct points comprising the scale are selected arbitrarily, the addition and multiplication constructions impart them with the special properties associated with 0, 1 and \( \infty \). However, from a projective standpoint, all points have identical properties. Most generally, the fundamental theorem of projective geometry, Schreier and Sperner [1935], states that in a space of dimension \( n \), there is a unique projective transformation between a pair of \( n+2 \) elements providing that for each part of the pair no \( n+1 \) elements belong to a space of dimension \( n-1 \) (see Section 2.3 for \( n=3 \)). For points on a line \( n=1 \) and thus three distinct points are related by a projective transformation. Three distinct new points may be chosen as another scale and all other points relabelled in terms of it. By way of projective transformations, all scales and subsequently all coordinates are projectively equivalent. Nevertheless, when a particular scale is employed, the scale points will still have the special properties of 0, 1 and \( \infty \) due to the definitions of the constructions. The approach taken here is to use the addition and multiplication constructions solely for the purpose of labelling the points. These constructions are not utilized further, such as to analytically express projective transformations in terms of nonhomogeneous coordinates. Thus, the special properties of the scale points do not enter into any further analytical development.

Projective homogeneous coordinates are introduced by associating a pair of numbers \( a_0, a_1 \), which can be written as the \( 2 \times 1 \) array \( [a_0 \ a_1]^T \), with the nonhomogenous coordinate of the point \( x_a \), such that

\[ a = \frac{a_1}{a_0}. \quad (4) \]

The homogeneous coordinates of \( x_a \) are not unique since \( [\lambda a_0 \ \lambda a_1]^T, \lambda \neq 0 \) also satisfies (4). The coordinates \( [0 \ \lambda]^T, \lambda \neq 0 \) are associated with the point \( x_\infty \). Thus to every pair of homogeneous coordinates, with the exception of \( [0 \ 0]^T \), there corresponds a unique point of the line and to every point of the line there corresponds a pair of coordinates, which to a scalar multiple, is unique.

Projective homogeneous coordinates for points in a plane are developed from the homogeneous coordinates of points in three distinct lines. As illustrated in Fig. 2.1.3, three noncollinear points are selected as the vertices of a triangle of reference and are used to establish two scale points on each side, \( w_0, w_\infty, x_0, x_\infty, \) and \( y_0, y_\infty \). A fourth point which does not lie on one of the edges is selected as the unit point. Lines through the unit point and each vertex establish on the sides opposite the unit points \( w_1, x_1 \) and \( y_1 \), thus completing the three scales. Together, the three vertices and the unit
point are said to form a reference frame and respectively their homogeneous coordinates are designated by

\[
\begin{align*}
[1 & 0 & 0]^T \\
[0 & 1 & 0]^T \\
[0 & 0 & 1]^T
\end{align*}
\]

and

\[
[1 & 1 & 1]^T. \tag{5}
\]

Every point on a side of the triangle may now be assigned a triple of homogeneous coordinates \([a_0, a_1, a_2]^T\). For example, on line \(U_y\) each of the scale points is characterized by \(a_2 = 0\). In terms of the homogeneous coordinates of \(U_y\), every point has the form \([a_0, a_1]^T\) which is now denoted in the planar system by \([a_0, a_1, 0]^T\). Analogously, points on \(U_w\) and \(U_x\) have coordinates of the form \([0, a_1, a_2]^T\) and \([a_0, 0, a_2]^T\).

For any point not on a side of the triangle, all of its coordinates \([a_0, a_1, a_2]^T\) will be nonzero. Let its projections on \(U_x\) and \(U_y\) from the opposite vertices be the points \(x_a\) and \(y_a\) with coordinates determined by their respective scales, \([x_{a0} 0 0]^T\) and \([y_{a0} 0 0]^T\). It is possible to select the coordinates of the point such that

\[
\begin{align*}
a_1 &= \frac{y_{a2}}{y_{a0}}, \quad a_2 &= \frac{x_{a2}}{x_{a0}}
\end{align*}
\]

that is in the ratios,

\[
a_0 : a_1 : a_2 = x_{a0} y_{a0} : x_{a0} y_{a1} : x_{a2} y_{a0}. \tag{7}
\]
The coordinates of $x_a$ and $y_a$ can thus be expressed respectively as $[a_0 \ a_1 \ a_2]^T$ and $[a_0 \ a_1 \ 0]^T$. For the system of coordinates to be consistent, it is necessary to demonstrate that the coordinates of $w_a$, the projection of point $a$ on $U_w$, are proportional to $[0 \ a_1 \ a_2]^T$. This can be proved using an argument involving cross ratios which is detailed in Veblen and Young [1910, pp. 175-176]. Therefore, to a nonzero scalar multiple, the projective homogeneous coordinates of a point on a plane are given by the triple $[a_0 \ a_1 \ a_2]^T$. It should be noted that the four points chosen for the reference frame, no three of which are collinear, are otherwise arbitrary. Thus any four such points may be chosen since by the fundamental theorem of projective geometry they are related by a unique projective transformation.

The method of using coordinates on three lines to establish coordinates on the plane leads to a recursive algorithm that may be used for a space of any dimension and is briefly described here for three dimensions.

Four noncoplanar points are chosen as the vertices of a tetrahedron of reference and a fifth point not incident with a face is selected as the unit point. As illustrated in Fig. 2.1.4, the four vertices and unit point are respectively assigned the coordinates,

$$
[1 \ 0 \ 0 \ 0]^T \\
[0 \ 1 \ 0 \ 0]^T \\
[0 \ 0 \ 1 \ 0]^T \\
[0 \ 0 \ 0 \ 1]^T 
$$

![Figure 2.1.4 Tetrahedron of reference for projective homogeneous coordinates in space.](image-url)
and

\[ [1 \ 1 \ 1 \ 1]^T. \]  

(8)

Lines through the unit point and each vertex establish unit points on the opposite sides respectively,

\[ [0 \ 1 \ 1 \ 1]^T \]
\[ [1 \ 0 \ 1 \ 1]^T \]
\[ [1 \ 1 \ 0 \ 1]^T \]
\[ [1 \ 1 \ 1 \ 0]^T. \]  

(9)

and together with the vertices, they are used to establish a planar reference system on each face.

Every point which is incident to a face of the tetrahedron may now be assigned a quadruple of homogeneous coordinates \([a_0 \ a_1 \ a_2 \ a_3]^T\). For example, on the face opposite the vertex \([0 \ 0 \ 0 \ 1]^T\), each of the points is characterized by \(a_3=0\). This face has precisely the same planar reference system (5) described previously and thus has the form \([a_0 \ a_1 \ a_2]^T\) which is now denoted in the spatial system by \([a_0 \ a_1 \ a_2 \ 0]^T\). Analogously, points on the remaining faces are characterized respectively by \(a_0=0\), \(a_1=0\), \(a_2=0\).

For any point not incident to a face of the tetrahedron, all of its coordinates \([a_0 \ a_1 \ a_2 \ a_3]^T\) are nonzero. It is necessary to show that the projections of the point onto the four faces can be assigned coordinates in the planar systems given respectively by

\[ [0 \ a_1 \ a_2 \ a_3]^T \]
\[ [a_0 \ 0 \ a_2 \ a_3]^T \]
\[ [a_0 \ a_1 \ 0 \ a_3]^T \]
\[ [a_0 \ a_1 \ a_2 \ 0]^T. \]  

(10)

This has been proved by Veblen and Young [1910, pp. 194-195]. It should be noted that the five points chosen for the reference frame, no four of which are coplanar, are otherwise arbitrary. Thus any such five points may be selected since by the fundamental theorem of projective geometry they are related by a unique projective transformation.

In the preceding developments, only the projective homogeneous coordinates of points in spaces of one, two and three dimensions were considered. One-dimensional spaces are also described by lines in a plane through a point or planes through a line and an analogous procedure may be developed to assign coordinates. Further, for two-dimensional spaces such as a lines in a plane or planes through a point and for three-dimensional spaces where for example the assemblage of all planes forms a three-space, coordinates may be assigned.

In the two-dimensional space of a plane, it is possible to simultaneously assign point and line coordinates and in three-space it is possible to assign point and plane coordinates without incurring any logical inconsistencies. In the next section such a methodology is presented and is then extended by determinant principles to assign coordinates to lines and screws in projective three-space.
Section 2.2 Homogeneous Coordinates

A point given by its coordinates and a point determined by its equation, or geometrically speaking by an infinite number of planes intersecting each other in that point, are quite different ideas, not to be confounded with one another. That is the case also with regard to a plane given by its coordinates and a plane represented by its equation, or considered as containing an infinite number of points. Hence is derived a double signification of a right line. It may be considered as the geometrical locus of points, or described by a point moving along it, and accordingly represented by two equations in \( x, y, z \) each representing a plane containing that line. But it may likewise be considered as the intersection of an infinite number of planes, or as enveloped by one of these planes, turning round it like an axis; accordingly it is represented by two equations in \( t, u, v \), each representing an arbitrary point of the line. The passage from one of the two conceptions to the other is a discontinuous one.

The geometrical constitution of space, hitherto referred either to points or to planes, may as well be referred to right lines. According to the double definition of such lines, there occurs to us a double construction of space.

In the first construction we imagine infinite space to be transversed by lines themselves consisting of points. An infinite number of such lines pass in all directions through any given point; each of these lines may be regarded as described by a moving point. This constitution of space is admitted when, in optics, we consider luminous points as sending out in all directions luminous rays, or, in mechanics, forces acting on points in every direction.

In the second construction infinite space is likewise regarded as transversed by right lines, but these lines are determined by means of planes passing through them. Every plane contains an infinite number of right lines having within it every position and direction, around each of which the plane may turn. We refer to this second conception when, in optics, we regard, instead of rays, the corresponding fronts of waves and their consecutive intersections, or when, in mechanics, according to POINSOT'S ingenious philosophical views, we introduce into its fundamental principles "couples," as well entitled to occupy their place as ordinary forces. The instantaneous axes of rotation are right lines of the second description. J. Plucker [1865, pp. 725-726]

The dualistic properties of projective geometry may be elegantly expressed in an analytic manner by employing homogeneous coordinates and determinant principles. The main objective of this section is the development of projective homogeneous coordinates in space, particularly line coordinates and their extensions to screws. However, it is appropriate to commence with point and line coordinates in the plane since this provides a foundation for the application of extensional principles to three-dimensional space, Forder [1940].

A line passing through two points may be described as the join of the two points and dually, the intersection point of two lines may be described as the meet of the two lines. Essentially, the principle of duality in the plane is that incidence relations remain valid when the roles of points and lines are interchanged (along with an appropriate alteration in terminology such as replacing join with meet). For example, the previous relations may be expressed as, a line (point) is the join (meet) of two points (lines).

In the plane, line coordinates are developed by first demonstrating, as in Coxeter [1942, pp. 78-80], that the condition for a point and a line to be incident may be expressed as the linear relation
\[ S_0 w_0 + S_1 w_1 + S_2 w_2 = 0 \]  
\( (1) \)

This relation can also be expressed more compactly by
\[ S^T w = 0 \]  
\( (2) \)

for which
\[ S = [S_0 \ S_1 \ S_2]^T, \quad w = [w_0 w_1 w_2]^T. \]  
\( (3) \)

The 3×1 arrays \( S \) and \( w \) are respectively projective homogeneous line and point coordinates in the plane and are unique to a nonzero scalar factor, i.e. only the ratios of the coordinates are significant. Assuming in (2) that the coordinates \( S \) have constant values while the coordinates \( w \) are free to vary, then (2) represents the locus of points which are incident to the line \( S \), or in other words, the equation of line \( S \). Dually, if the coordinates \( w \) are assumed constant while the coordinates \( S \) are free to vary, then (2) represents the pencil of lines which are incident to the point \( w \), or in other words, the equation of point \( w \).

Equation (2) may be applied to express line coordinates in terms of point coordinates and reciprocally. Let \( x \) and \( y \) be two given distinct points on \( S \) and let \( w \) be any variable point on the line. Then using (2) yields three equations
\[ [w \times y]^T S = 0 \]  
\( (4) \)

for which the vanishing of the determinant
\[ |w \times y| = 0 \]  
\( (5) \)

yields an equation in \( w \).
\[ w_0 |x_1 y_2| + w_1 |x_2 y_0| + w_2 |x_0 y_1| = 0 \]  
\( (6) \)

where
\[ |x_i y_j| = x_i y_j - x_j y_i \]  
\( (7) \)

is the \( ij \) minor in (5). Based on the expansion (6), it is useful to define the determinants of the nonsquare arrays \([w]\) (or \( w \)) and \([xy]\) as the 3×1 column arrays
\[ [w] = [w_0 \ w_1 \ w_2]^T \]  
\( (8) \)
\[ [xy] = [ |x_1 y_2| \ |x_2 y_0| \ |x_0 y_1| ]^T. \]  
\( (9) \)

Thus equations (5) or (6) may be expressed simply as
\[ [w]^T [xy] = 0 \quad \text{or} \quad w^T [xy] = 0. \]  
\( (10) \)

Comparing (10) with (2) yields the coordinates of line \( S \) to a scalar multiple in terms of points \( x \) and \( y \),
\[ S = |xy|. \]  
\( (11) \)

Since \( x \) and \( y \) are assumed distinct, then from (5) it is readily deduced that any point on the line \( S \) may be expressed in the form
\[ w = \alpha x + \beta y \]  
\( (12) \)

where scalars \( \alpha \) and \( \beta \) are both not simultaneously zero.

Equation (12) is referred to as the freedom equation of the line and any point on it is given by the ratio \( \alpha:\beta \). Using
it may be shown that the coordinates of \( S \) are, to a scalar factor, independent of which two distinct points are chosen. For example, replacing \( x \) and \( y \) with \( (\alpha_0 x + \beta_0 y) \) and \( (\alpha_1 x + \beta_1 y) \) in (11) yields

\[
S = |\alpha \beta| \cdot |xy|
\]

(13)

where the scalar \(|\alpha \beta| \neq 0\) since the points are assumed distinct.

Because of the duality principle, the development of line coordinates in terms of point coordinates is completely analogous to the preceding analysis. Let \( T \) and \( U \) be two given distinct lines incident to \( w \), then if the variable line \( S \) is also concurrent,

\[
[S \, T \, U]^T \, w = 0
\]

(14)

for which

\[
|S \, T \, U| = 0
\]

(15)

yields an equation in \( S \),

\[
S_0 |T_1 \, U_2| + S_1 |T_2 \, U_0| + S_2 |T_0 \, U_1| = 0
\]

(16)

or equivalently,

\[
|S| \cdot |T \, U| = 0.
\]

(17)

Comparing (17) with (12) yields the coordinates of point \( w \) to a scalar multiple in terms of lines \( T \) and \( U \),

\[
w = |T \, U|.
\]

(18)

Analogous to (12), the freedom equation for any line incident to \( w \) is deduced from (15)

\[
S = \alpha T + \beta U
\]

(19)

where \( \alpha \) and \( \beta \) are both not simultaneously zero.

In Section 2.1 the vertices of the reference triangle were assigned point coordinates corresponding to the columns of \( I_3 \) (where \( I_n \) is the \( n \times n \) identity matrix) and the first column represented what is usually called the origin. By substituting pairs of vertices in (11) it is shown that the line coordinates of any side and the point coordinates of the opposite vertex are given by the same column of \( I_3 \). The side opposite the origin is often called the line at infinity though no reference to distance is implied here.

For the plane, point and line coordinates can be summarized using

\[
|x|, \quad |T|
\]

(20)

\[
|x \, y|, \quad |T \, U|.
\]

(21)

In (20), point and line coordinates are given respectively by the three \( 1 \times 1 \) determinants of \([x]\) and \([T]\) selected in the order 0, 1, 2. In (21) line and point coordinates are given respectively by the three \( 2 \times 2 \) determinants of \([xy]\) and \([TU]\) selected in the order 12, 20, 01.

In higher order spaces many of the developments for the plane may be further generalized by application of extensional determinant principles which were first introduced by Grassmann, see Forder [1940], Klein [1908]. For three-dimensional...
projective space, the point may be chosen as the fundamental element and then a line is the join of two points and a plane is the join of three points. Dually, the plane may be selected as the fundamental element and then a line is the meet of two planes and the point is the meet of three planes. In space, the point and plane are dual elements whereas the line is self-dual. The duality principle in space asserts that incidence relations that are valid for points, lines and planes remain valid when their roles are interchanged with planes, lines and points along with an appropriate alteration in terminology.

The condition for a point and plane to be incident may be expressed as the linear relation

\[ S_0 w_0 + S_1 w_1 + S_2 w_2 + S_3 w_3 = 0 \]  
(22)

or equivalently

\[ S^T w = 0 \]  
(23)

for which now

\[ S = [S_0 S_1 S_2 S_3]^T, \quad w = [w_0 w_1 w_2 w_3]^T. \]  
(24)

The 4×4 arrays \( S \) and \( w \) are respectively projective homogeneous plane and point coordinates and are unique to a nonzero scalar factor since only the ratios of the coordinates are significant. Equation (23) may represent either the locus of points incident to plane \( S \) or dually, the bundle of planes incident to point \( w \), the difference being respectively whether \( S \) or \( w \) is assumed fixed while the other is free to vary.

Plane coordinates may be developed from point coordinates by requiring that one variable and three given noncollinear points be incident to a plane

\[ [w \times y \ z]^T S = 0 \]  
(25)

for which

\[ |w \times y \ z| = 0 \]  
(26)

yields an equation in the variable point \( w \)

\[ w_0 |x_1 y_2 z_3| + w_1 |x_2 y_0 z_3| + w_2 |x_0 y_1 z_3| + w_3 |x_1 y_0 z_2| = 0 \]  
(27)

and \( |x_{ij} y_j z_k| \) is the \( ijk \) minor in (26). Alternately, (27) may be expressed more compactly using nonsquare determinants

\[ |w|^T |xyz| = 0 \text{ or } w^T |xyz| = 0 \]  
(28)

where

\[ |w| = [w_0 \ w_1 \ w_2 \ w_3]^T \]  
(29)

\[ |xyz| = [|x_1 y_2 z_3| \ |x_2 y_0 z_3| \ |x_0 y_1 z_3| \ |x_1 y_0 z_2|]^T \]  
(30)

Comparing (30) with (23) yields the coordinates of plane \( S \) to a scalar factor in terms of the points \( x, y \) and \( z \),

\[ S = |x \ y \ z|. \]  
(31)

Since \( x, y \) and \( z \) are noncollinear, then from (26) it is deduced that any point on the plane \( S \) may be expressed in the form,
\[ w = \alpha x + \beta y + \gamma z \]  
(32)

where the scalars \( \alpha \), \( \beta \) and \( \gamma \) are all not simultaneously zero.

Equation (32) is the freedom equation of the plane and any point on it is specified by the ratios \( \alpha : \beta : \gamma \). Substituting for \( x \), \( y \) and \( z \) in (31), any other three noncollinear points on the plane \( S \), \( \alpha_1 x + \beta_1 y + \gamma_1 z \), \( i = 1, 2, 3 \) yields

\[ S = |\alpha \beta \gamma| \cdot |xyz| \]  
(33)

where the scalar \( |\alpha \beta \gamma| \neq 0 \) since the points are noncollinear.

Briefly, by the principle of duality, the development of plane coordinates from point coordinates is entirely analogous to the preceding development. Let one variable plane and three given planes, which themselves do not meet in a line, all be concurrent at a point.

\[ [S \quad T \quad U \quad V]^T w = 0 \]  
(34)

for which

\[ |S \quad T \quad U \quad V| = 0 \]  
(35)

yields an equation in the variable plane \( S \),

\[ S_0 |T_1 U_2 V_3| + S_1 |T_2 U_3 V_3| + S_2 |T_0 U_2 V_3| + S_3 |T_1 U_0 V_2| = 0 \]  
(36)

or equivalently

\[ |S|^T |T \quad U \quad V| = 0. \]  
(37)

Comparing (37) with (38) yields the coordinates of point \( w \) to a scalar factor in terms of planes \( T, U \) and \( V \),

\[ w = |T \quad U \quad V| \]  
(38)

Further, the freedom equation of the planes through \( w \) is deduced from (35),

\[ S = \alpha T + \beta U + \gamma V \]  
(39)

for \( \alpha \), \( \beta \) and \( \gamma \) not all simultaneously zero.

In Section 2.1 the vertices of the reference tetrahedron were assigned coordinates corresponding to the columns of \( I_4 \).

By substituting triples of vertices in (31) it may be shown that the plane coordinates of any face and the point coordinates of the opposite vertex are given by the same column of \( I_4 \). For instance, the first column corresponds to the vertex at the origin and to the face that is often referred to as the plane at infinity.

In space, point and plane coordinates may be summarized and line coordinates introduced using the nonsquare determinants,

\[ |x|, \quad |T| \]  
(40)

\[ |x \ y|, \quad |T \ U| \]  
(41)

\[ |x \ y \ z|, \quad |T \ U \ V|. \]  
(42)

In (40) point and plane coordinates are given respectively by the four 1x1 determinants of \([x]\) and \([T]\) selected in the order 0, 1, 2, 3. For (42), plane and point coordinates are
given respectively by the four \(3 \times 3\) determinants of \([x, y, z]\) and 
\([T, U, V]\) selected in the order 123, 203, 013, 102.

Plucker's ray and axis line coordinates, Plucker [1865, 
1866], are defined by (41) respectively as the six \(2 \times 2\) determinants of 
\([x, y]\) and \([T, U]\) selected in the order 01, 02, 03, 23,
31, 12. Ray line coordinates \(p\) represent the join of two 
points \(x, y\) and axis coordinates \(P\) represent the meet of two 
planes and

\[
P = [p_{10} p_{01} p_{02} p_{31} p_{32} p_{21}]^T, \quad p_{1j} = |x_j y_j| \quad (43)
\]

\[
P = [p_{01} p_{02} p_{03} p_{23} p_{31} p_{12}]^T, \quad p_{1j} = |T_j U_j| \quad (44)
\]

or more briefly,

\[
p = |x \ y| \quad , \quad P = |T \ U| \quad . \quad (45)
\]

Referring to Fig. 2.2.1, the relationship between ray and 
axis coordinates is derived from the incidence relations of 
two points with two planes,

\[
T^T x = 0 \quad (46)
\]

\[
T^T y = 0 \quad (47)
\]

\[
U^T x = 0 \quad (48)
\]

\[
U^T y = 0 \quad . \quad (49)
\]

Forming in turn, \(T \cdot (48) - U \cdot (46), T \cdot (49) - U \cdot (47),\)
x \cdot (47) - y \cdot (46) and x \cdot (49) - y \cdot (48) yields respectively,
\[
[p^*]x = 0
\tag{50}
\]
\[
[p^*]y = 0
\tag{51}
\]
\[
[p^*]T = 0
\tag{52}
\]
\[
[p^*]U = 0
\tag{53}
\]

where the rank two skew-symmetric arrays are given by

\[
[p^*] =
\begin{bmatrix}
  P_{01} & P_{02} & P_{03} \\
  P_{10} & P_{12} & P_{13} \\
  P_{20} & P_{21} & P_{23} \\
  P_{30} & P_{31} & P_{32}
\end{bmatrix}
\tag{54}
\]

\[
[p^*] =
\begin{bmatrix}
  P_{01} & P_{02} & P_{03} \\
  P_{10} & P_{12} & P_{13} \\
  P_{20} & P_{21} & P_{23} \\
  P_{30} & P_{31} & P_{32}
\end{bmatrix}
\tag{55}
\]

From (50), (51) and (54), the rows (or columns) of \([p^*]\) are four planes through the line, each incident with a vertex of the reference tetrahedron. From (52), (53) and (55), the rows (or columns) of \([p^*]\) are four points on the line, each incident with a face of the reference tetrahedron.

Forming either (50) \(- y^T \cdot (51) \cdot x^T\) or (52) \(- U^T \cdot (53) \cdot T^T\) yields the matrix equation

\[
[p^*][p^*] = 0
\tag{56}
\]
In Table 2.2.1, (56) has been expanded into components to yield (57). Then the off-diagonal terms are used to form the twelve equations in (58) written as a 4 × 4 array. Comparing in turn five equations in (58) given by the positions 30, 10, 13, 03, 01 yields

\[
\begin{align*}
P_{23} &= P_{31} = P_{12} = P_{01} = P_{02} = P_{03} = \mu \\
P_{01} &= P_{02} = P_{03} = \mu \\
P_{23} &= P_{31} = P_{12} = \mu
\end{align*}
\]  

(59)

where \( \mu \) is a nonzero scalar.

It should be noted that it is not necessary to explicitly include \( \mu \) since it is included implicitly by only assigning significance to the ratios of homogeneous coordinates. Thus setting \( \mu = 1 \) and arranging (59) in matrix form yields

\[
\begin{align*}
P &= \tilde{p} \\
p &= \tilde{\Delta} \tilde{p}
\end{align*}
\]  

(60)

(61)

where

\[
\begin{bmatrix}
\tilde{\Delta} & I_3 \\
I_3 & \tilde{\Delta}
\end{bmatrix}
\]  

(62)

\[
\tilde{\Delta}_{\text{T}} = \tilde{\Delta}, \quad \tilde{\Delta} \tilde{\Delta} = I_6
\]  

(63)

Since in (60) and (61), \( p \) and \( P \) are derived from dual elements, the induced linear transformation \( \tilde{\Delta} \) is a correlation which is signified by tilda. The existence of this correlation is due to the fact that in three-dimensional space lines are self-dual elements and thus \( \tilde{\Delta} \) represents the identical correlation of lines. The matrix \( \tilde{\Delta} \) represents a very simple method of transforming between ray and axis coordinates, the first and last three components are merely exchanged. Since a double application of the exchange yields back the initial values, then as expressed by (63), \( \tilde{\Delta} \) is clearly an involution matrix.

In ray coordinates, the condition for two lines to be incident may be obtained by letting \( p \) be the join of \( w, x \) and \( q \) be the join of \( yz \). When the lines intersect then all four points are coplanar and

\[
[w \times y \times z] = 0
\]  

repeated, (26)

which is expanded by the first two columns to yield

\[
p_{\text{T}} \tilde{\Delta} q = 0 \quad \text{or} \quad |w \times x|_{\text{T}} \tilde{\Delta} |y \times z| = 0.
\]  

(64)

Alternatively, in axis coordinates let \( P \) and \( Q \) be the same two lines where \( P \) is the meet of \( X, T \) and \( Q \) is the meet of \( U, V \). When the lines intersect then all four planes are concurrent at a point and the expansion of

\[
[S \ T \ U \ V] = 0
\]  

repeated, (34)

yields

\[
p_{\text{T}} \tilde{\Delta} Q = 0 \quad \text{or} \quad |S \ T|_{\text{T}} \tilde{\Delta} |U \ V| = 0.
\]  

(65)

Using (60), (61) with (64), (65) gives alternative expressions for intersection,
\[ p^T q = 0 \quad \text{or} \quad |w x|^T |U V| = 0 \quad \quad (66) \]
\[ p^T q = 0 \quad \text{or} \quad |S T|^T |x y| = 0. \quad \quad (67) \]

Because dual coordinates are used in (66), (67) their form is analogous to (23) which expresses the incidence of a point and a plane.

Since only the ratios of homogeneous coordinates are significant, the six coordinates of a line represent five parameters. However, only four independent parameters are required to specify a line in space and therefore six coordinates must be related by a single equation. In terms of ray and axis coordinates, this relation is obtained by expanding the singular determinants

\[ |x y x y| = 0 \quad , \quad |U T U T| = 0 \quad \quad (68) \]

to yield

\[ p^T \tilde{A} p = 0 \quad , \quad p^T \tilde{A} p = 0 \quad \quad (69) \]

or equivalently

\[ p^T p = p^T p = 0. \quad \quad (70) \]

It is interesting to examine the line coordinates corresponding to the tetrahedron of reference. Referring to Fig. 2.2.2, the vertices along with their opposing faces are both labelled 0, 1, 2, 3 and their coordinates correspond in order to the columns of \( I_r \). Forming the ray coordinates of the six joins of vertex pairs and the six meets of face

Figure 2.2.2 The tetrahedron of reference labelled with a) point coordinates and ray coordinates, b) plane coordinates and axis coordinates.
pairs both in the order 01, 02, 03, 23, 31, 12, then the resulting coordinates of both sets correspond in order to the columns of $I_6$. Opposite edges of the tetrahedron (i.e. nonintersecting), one expressed in ray coordinates and the other in axis coordinates, correspond to the same column of $I_6$.

Unlike point and plane coordinates, there are in general no freedom equations for lines using line coordinates corresponding to (32) and (39). (There are however freedom equations for lines in terms of either two points or two planes.) For example, let $q, r$ be two lines and $u, \lambda$ be two scalars and consider

$$ p = u q + \lambda r. \quad (71) $$

For $p$ to be a line it must satisfy the quadratic form of (69).

However,

$$ p^T \tilde{A} p = 2 u \lambda q^T \tilde{A} r \quad (72) $$

which only vanishes when $q$ and $r$ intersect.

In general, a linear combination of lines in either ray or axis coordinates is defined here as a screw, which includes lines as special cases. Since the homogeneous coordinates of a screw need not satisfy (69), it follows that there are $5$ screws in space. When two screws satisfy any of the equivalent relations

$$ p^T \tilde{A} q = 0, \quad p^T \tilde{A} Q = 0 \quad (73) $$

$$ p^T Q = 0, \quad p^T q = 0 \quad (74) $$

the screws are said to be reciprocal, a property which is analogous to incidence.

For a general linear combination of $n$ screws (or lines), $1 \leq n \leq 6$, the freedom equations

$$ p = \lambda_1 q + \ldots + \lambda_n r $$

$$ P = \lambda_1 Q + \ldots + \lambda_n R $$

where $\lambda_1 \ldots \lambda_n$ are all not simultaneously zero, are said to describe an $n$-system of screws, Ball [1900], Hunt [1978].

Screws may also be viewed abstractly as points in a five-dimensional space as suggested by F. Klein (see Jessop [1903]). Lines in this space lie on a surface which is given by (69). Dually, hyperplanes in this space correspond to five-systems of screws, which also may be described by six homogeneous coordinates. By generalizing the development of coordinates presented here, systems of coordinates may be systematically derived for $n$-systems, $1 \leq n \leq 5$. Detailed references on the generation of extensional systems of coordinates based on determinant properties are given by Forder [1940], Sommerville [1929], Hodge and Pedoe [1947, 1952], but will not be considered here further.
Section 2.3 Projective Transformations

... I shall enunciate two general principles which I have habitually emphasized and have put into the foreground in these fundamental geometric discussions. Although in this generality they sound at first somewhat obscure, they will, with concrete illustrations, soon become clear. One of them is that the geometric properties of any figures must be expressible in formulas which are not changed when one changes the coordinate system, i.e. when one subjects all the points of the figure simultaneously to one of our transformations; and, conversely, any formula which, in this sense, is invariant under the group of these coordinate transformations must represent a geometric property. As simplest examples, which all of you know, let me remind you of the expression for the distance or for the angle, in the figure of two points or of two lines. We shall have to do repeatedly with these and with many other similar formulas in the following pages. For the sake of clearness, I shall give a trivial example of non-invariant formulas: The equation $y = 0$, for the figure consisting of the point $(x, y)$ of the plane, says that this point lies on the x axis, which is, after all, a thoroughly unessential fact, foreign to the nature of the figure, useful only in serving to describe it. Likewise, every non-invariant equation represents some relation of the figure to external, arbitrarily added, things, in particular to the coordinate system, but it does not represent any geometric property of the figure.

The second principle has to do with a system of analytic magnitudes which are formed from the coordinates of points $1, 2, \ldots$, such as $X, Y$, and $N$, for example. If this system has the property of transforming into itself, in a definite way, under a transformation of coordinates, i.e., if the system of magnitudes formed from the new coordinates of the points $1, 2, \ldots$, expresses itself in terms exclusively of these magnitudes formed in the same way from the old coordinates (the coordinates themselves not appearing explicitly), then we say that the system defines a new geometric configuration, i.e., one which is independent of the coordinate system. In fact, we shall classify all analytic expressions according to their behavior under coordinate transformation, and we shall define as geometrically equivalent two series of expressions which transform in the same way. Felix Klein [1908, pp. 25-26]

This section examines linear projective transformations of homogeneous coordinates. Klein [1872] has enunciated a definition of geometry which, except for minor extensions, is still very applicable today. Essentially, Klein stated that a geometry is defined as the properties of a space which remain invariant under all transformations of space (or the coordinate system) by a group of transformations.

For projective geometry, the group of transformations is characterized by those which preserve relations of incidence. Commencing with the group of projective point collineations, the corresponding induced collineations for planes, lines and screws are developed with respect to an elegant tetrahedronal principle employing determinant relations. Using a simple device, many of the results for collineations are extended to the nongroup of correlations. An analysis of projective transformations not only identifies important invariant relations but also forms a foundation for developing metrical geometries in Chapter 3.

A collineation is a one-to-one linear transformation in which each element of space is mapped into a corresponding element of the same type (e.g. point to point) whereas a correlation differs in that each element is mapped into a corresponding dual element (e.g. point to plane). A projective transformation is uniquely determined by five pairs of corresponding points in space provided that no four of the five points in either pair are coplanar. For the collineation,
\[ ut = Kx \]  

(1)

where \( \mu \) is included explicitly as a factor of proportionality and where the \( 4 \times 4 \) matrix \( K \) is given by

\[
K = \begin{bmatrix} A & B & C & D \end{bmatrix}^T.
\]

(2)

Since only the ratios of homogeneous coordinates are significant, the four equations in (1) can be reduced to three ratios of equations by, for example, dividing the last three equations by the first equation and thus the explicit factor \( \mu \) is eliminated. Multiplying out the ratios and expressing them in matrix form yields

\[
\begin{bmatrix} -t_1 x^T & t_0 x^T \\
-t_2 x^T & t_0 x^T \\
-t_3 x^T & t_0 x^T \\
\end{bmatrix} \begin{bmatrix} A \\
B \\
C \\
D \\
\end{bmatrix} = [0] \]

(3)

where \( t_0 \ldots t_3 \) are the coordinates of \( t \) and where the \( 3 \times 16 \) matrix multiplies the \( 16 \times 1 \) column array containing the unknown coefficients. Substitution for \( t \) and \( x \) by five pairs of corresponding points yields 15 homogeneous equations which are sufficient to solve for 15 ratios involving the elements of \( K \). Thus the projective collineation is uniquely determined to a scalar factor and \( K \) is nonsingular since the mapping is one-to-one.

In (1) the factor \( \mu \) was explicitly included to facilitate in the solution for \( K \). However, it is convenient to absorb the factor by substituting \( \mu = 1 \) which is permissible provided it is understood that only the ratios are significant,

\[ t = Kx. \]

(4)

A projective collineation of points also induces a projective collineation of planes which may be determined using incidence properties. Let \( x \) be incident to plane \( T \),

\[ T^T x = 0. \]

(5)
The induced transformation \( k \) maps \( T \) into another plane \( X \)

\[ X = kT \]

(6)
such that incidence is preserved,

\[ X^T t = 0. \]

(7)
Substituting (4), (6) in (7),

\[ T^T (k^T k) x = 0 \]

(8)
and comparing with (5) yields

\[ k = K^{-T} \]

(9)
to an arbitrary nonzero scalar multiple. Matrix \( k \) can be calculated by replacing each element of \( K \) with its cofactor (signed minor), and dividing by the scalar \( |K| = |A B C D| \) (although this last step is not essential),

\[ k = \left[ \begin{array}{cccc} B C D & C A D & A B D & B A C \end{array} \right] \left[ A B C D \right]^T / |A B C D| \]

(10)
or more simply as

\[ k = [a \ b \ c \ d]^T. \] (11)

The four 4×3 determinants in (10) have been formed from ABCD in the order 123, 203, 013, 102 which is also the same order used in expanding each of these nonsquare determinants into components, see (2.2.42).

There is a useful geometric interpretation for \( K \) and \( k \). Let ABCD represent the coordinates of four planes whose equations can be written as

\[ [A \ B \ C \ D]^T w = 0. \] (12)

Since \( K \) is nonsingular then (12) has no solution other than \( w=0 \) which does not represent a point in homogeneous coordinates. Thus the four planes do not have a common point and they therefore form a tetrahedron. In (10), each row is the meet of three planes and is thus a vertex of the same tetrahedron. The vertices abcd are respectively opposite the faces ABCD since (2), (9) and (11) yield the incidence relations

\[ [A \ B \ C \ D]^T [a \ b \ c \ d] = I_4. \] (13)

Additionally, \( K \) can be expressed in terms of \( k \) and by analogy with (10),

\[ K = \left[ \begin{array}{cccc} b & c & a & d \\ a & b & c & d \\ b & a & c & d \\ c & b & a & d \end{array} \right]^T / |a b c d|. \] (14)

The collineation of points not only induces a collineation of planes, but also induces a collineation of lines. Let \( x, y \) and \( t, u \) be a pair of corresponding points

\[ t = Kx, u = Ky. \] (15)

Forming the join of each pair, then the lines are expressed in ray coordinates by

\[ p = |xy|, q = |tu| \] (16)

and the line \( p \) is transformed into the line \( q \). Substituting (15) into (16) yields

\[ q = |Kx Ky| \] (17)

and then substituting (2) in (17) gives the nonsquare determinant

\[ q = \begin{vmatrix} A^T x & A^T y \\ B^T x & B^T y \\ C^T x & C^T y \\ D^T x & D^T y \end{vmatrix}. \] (18)

The first coordinate of \( q \) is given by

\[ q_{01} = A^T x B^T y - A^T y B^T x \]

\[ = A^T [xy^T - yx^T] B \]

\[ = A^T [p^*] B \] (19)

where \([p^*]\) is the skew-symmetric matrix given in (2.2.55) where elements are \( p_{ij} = |x_i y_j| \). Expanding the bilinear
expression in (19) yields after some manipulations,

\[ q_{01} = |AB|^T p. \]  

The remaining components of \( q \) are determined by analogy with \( q_{01} \) which yields

\[ q = [|AB| |AC| |AD| |CD| |DB| |BC|]^T p \]  

and which is more concisely expressed by

\[ q = \hat{K} p. \]  

The six \( 4 \times 2 \) determinants in (21) have been formed from the planes \( ABCD \) in the order \( 01, 02, 03, 23, 31, 12 \) which is also the same order used in expanding each of the nonsquare determinants into components, see (2.2.41).

Analogously, the induced collineation for axis coordinates can be developed from a pair of corresponding planes,

\[ X = kT, \ Y = kU. \]  

Forming the meet of each pair in axis coordinates,

\[ L = |XY| \quad M = |TU| \]  

then the collineation transforms line \( L \) into line \( M \) and using (23) in (24) yields

\[ M = |kT \, kU|. \]  

Expanding the terms in (24a) and using (11) gives a result which is analogous to (21),

\[ M = [|ab| |ac| |ad| |cd| |db| |bc|]^T L \]  

or more concisely,

\[ M = \hat{K} L. \]  

Since a collineation preserves incidence properties, then if \( p \) and \( L \) are incident then so are \( q \) and \( M \),

\[ L_T p = 0 \]  

\[ M_T q = 0. \]  

Substitution of (22), (26) in (28) yields

\[ L_T (\hat{K}^T K) p = 0 \]  

which leads to a result analogous to (9),

\[ \hat{k} = \hat{K}^{-T} \]  

upon comparing (27) and (29) for general intersecting lines \( L, p \). Although (30) is correct to a scalar multiple, it can be shown that it is actually an algebraic identity when the plane collineation is given by (2) and (14) and the point collineation is given by (10) and (11).

Figure 2.3.1.1 is used to illustrate the geometric interpretations of the induced ray and axis line collineations. From (21), the collineation of ray coordinates is given by \( \hat{K} \) whose rows are the axis coordinates for the six edges of the illustrated tetrahedron. From (25), the collineation of
axis coordinates is given by \( \hat{k} \) whose rows are the ray coordinates for the six edges of the tetrahedron. Thus the same tetrahedron is intimately related to the four collineations \( K, k, \hat{K}, \hat{k} \).

This tetrahedron relationship leads to two important algebraic identities for the induced line collineations,

\[
\hat{K}^T \Delta \hat{K} = |A B C D| \Delta \tag{31}
\]

\[
\hat{k}^T \Delta \hat{k} = |a b c d| \Delta. \tag{32}
\]

where

\[
\Delta = \begin{bmatrix}
I_3 \\
I_3
\end{bmatrix}.
\tag{2.2.62}
\]

Briefly, the identities (31), (32) are the conditions for the rows of \( \hat{K} \) and \( \hat{k} \) to be edges of the tetrahedron in Fig. 2.3.1. These identities are demonstrated by first substituting (30) into (31), (32) to yield

\[
|a b c d| \Delta = \hat{K}^T \Delta \hat{K}
\tag{33}
\]

\[
|A B C D| \Delta = \hat{K}^T \hat{K}
\tag{34}
\]

and where additionally

\[
|a b c d| \cdot |A B C D| = 1.
\tag{35}
\]

Relations (33), (34) are easily proved by substituting in the values of \( \hat{k} \) and \( \hat{K} \) in (21) and (25) and by noting that,
\[ |ab|^T \Delta | cd| = |a\ b\ c\ d| \quad (36) \]
\[ |AB|^T \Delta | CD| = |A\ B\ C\ D| \quad (37) \]

since the left sides are merely the Laplacian expansions of the right sides by the first two columns. Any other product such as
\[ |ab|^T \Delta | ad| = |a\ b\ a\ d| = 0 \quad (38) \]

clearly vanishes since two columns are identical.

Using incidence properties, (31) and (32) are now verified to a scalar multiple. Since \( p, q, L, M \) are lines then they satisfy the identical relations given by (2.2.69),
\[ p^T \Delta p = 0 \quad L^T \Delta L = 0 \quad (39) \]
\[ q^T \Delta q = 0 \quad M^T \Delta M = 0 . \quad (40) \]

Substituting (22), (26) into (40) yields
\[ p^T (K^T \Delta K)p = 0 \quad L^T (k^T \Delta k)L = 0 . \quad (41) \]

Comparing (39) and (41) for independent \( p \) and \( L \) yields (31), (32) to a nonzero scalar multiple,
\[ K^T \Delta K = \mu \Delta \quad k^T \Delta k = \mu \Delta . \quad (42) \]

Equation (42) (or (31), (32)) is important since it yields the conditions which are necessary for a 6\times6 matrix to represent an induced projective collineation of lines.

A 6\times6 matrix contains 36 elements and since only the ratios are significant there are 35 independent parameters. In (42), either of the equations represents only 21 different scalar equations since the matrix equations are symmetric. Because these equations are not all homogeneous, the scalar multiplier \( \mu \) is eliminated by considering the 20 ratios of equations (in a manner analogous to (3)). Thus the 35 parameters are related by 20 constraint equations to yield 35-20=15 independent parameters to describe an induced projective collineation of lines. This is in agreement with the projective point and plane collineations whose associated 4\times4 matrices yield 15 independent ratios.

A linear combination of lines, as previously defined by
\[ p = \lambda_1 q + \ldots \lambda_n r \quad (2.2.75) \]
\[ P = \lambda_1 Q + \ldots \lambda_n R \quad (2.2.76) \]
is in general a screw. The induced projective collineation of screws in space is identical to that for lines by way of linearity,
\[ \hat{K}p = \lambda_1 \hat{K}q + \ldots \lambda_n \hat{K}r , \quad \hat{k}P = \lambda_1 \hat{k}Q + \ldots \lambda_n \hat{k}R . \quad (43) \]

If a 6\times6 matrix does not satisfy (42) then it is not an induced projective collineation of lines. It is possible for this to occur in two ways. Firstly, if the rows of the matrix all represent lines then they cannot form a tetrahedron. Secondly, if any row is a screw then the matrix is not an
induced collineation. General transformations of this type have been investigated by Ball [1900], which he called homographic transformations. It should be noted that in the literature the terms homographic transformations and projective transformations are often used synonymously but are used distinctively here. Effectively, Ball treated screws as points in a projective five-space and therefore a homographic collineation is the most general one-to-one linear transformation of these points.

It is not difficult to verify that nonsingular projective point and plane collineations each form a group of transformations under the operation of composition or matrix multiplication. Nonsingular induced projective line collineations also form a group of transformations and it is useful to demonstrate the property of closure. Using ray coordinate transformations let \( \hat{K}, \hat{J} \) be two nonsingular collineations and it is necessary to show that \( \hat{KJ} \) is also such a collineation, i.e. it satisfies (42),

\[
(KJ)^T \Delta (KJ) = J^T (\hat{K})^T \Delta \hat{K} J
= \mu J^T \Delta \hat{J}
= \mu \Lambda
\]

where \( \mu \Lambda \) is a nonzero scalar multiplier.

Since induced collineations constitute a group, then it follows from (42) that the bilinear forms,

\[
p^T \Delta q, \ p^T \Delta Q
\]

and in particular the quadratic forms

\[
p^T \Delta p, \ p^T \Delta p
\]

are invariant expressions with respect to induced collineations. From linear principles, this is true whether \( p, q \) (\( P, Q \)) are lines or screws. The forms (45) are often referred to as the mutual moment of two lines or screws although no metrical connotation is implied here. When screws are considered as points in a five-space then the lines of three-space are represented as points on the quadric surface

\[
p^T \Delta p = 0, \ p^T \Delta p = 0
\]

which is sometimes called a Grassmannian. Hodge and Pedoe [1952]. In the group of homographic transformations of screws (i.e. nonsingular 6×6 matrices), induced projective line transformations constitute a subgroup which leaves the quadric (47) invariant (transforms into itself).

Projective transformations are classified as either collineations or correlations. Correlations are linear one-to-one transformations which map each element into a dual element. Since the product of two correlations is a collineation, it follows that correlations do not possess the property of closure and thus do not form a group. However, collectively collineations and correlations form the group of projective transformations.
In the development of correlations it is useful to first define two distinct spaces where elements in one are denoted by a prime. Consider the incidence of a point \( x \) on a plane \( T \),

\[
T^T x = 0 \quad (47)
\]

and the correlation,

\[
x' = \Lambda x, \quad t' = \Lambda T. \quad (48)
\]

Since projective transformations must preserve incidence relations then,

\[
t'^T x' = T^T (\Lambda^T \Lambda)x = 0 \quad (49)
\]

and by comparing (47) and (49) then to a scalar multiple,

\[
\Lambda = \Lambda^T \quad (50)
\]

The inverse transformations between the two spaces are given by inverting (48),

\[
T = \Lambda^{-1} t', \quad x = \Lambda^{-1} x' \quad (51)
\]

or equivalently using (50),

\[
T = \Lambda^T t', \quad x = \Lambda^T x'. \quad (52)
\]

Figure 2.3.2 illustrates the mapping between the two spaces described by (48) and (52). It is noted that the two point
to plane transformations (solid lines) are transposes of each other as are the two plane to point transformations (dotted lines).

Correlations of a single space onto itself may be deduced by allowing the two distinct spaces to coincide. For the correlation to be well-defined it is necessary that

$$\tilde{\mu} = \tilde{\lambda}^T, \quad \tilde{\mu} = \tilde{\lambda}^T$$

(53)

for which either $\mu = -1$ and $\tilde{\lambda}, \lambda$ are skew-symmetric or $\mu = 1$ and $\tilde{\lambda}, \lambda$ are symmetric. Skew-symmetric correlations are called null polarities and since the matrix is of an even order it is generally nonsingular. Null polarities have many interesting properties, especially in relation to the linear complex, Jessop [1903], Busemann and Kelly [1953]. Symmetric correlations are referred to as polarities and are used to establish metrics in Section 3.1. Generally, the only correlations that are employed here subsequently are polarities.

The development of polarities (or more generally correlations) is facilitated by introducing the polarity $\tilde{I}_n$,

$$\tilde{I}_n \tilde{I}_n = \tilde{I}_n$$

(54)

where $\tilde{I}_n$ is the $n \times n$ identity collineation. A polarity may be expressed as a product of $\tilde{I}_n$ and a symmetrical collineation $k, k, k,$

$$\tilde{\lambda} = \tilde{\lambda}_4 k = k \tilde{I}_4$$

$$\tilde{\lambda} = \tilde{\lambda}_4 K = K \tilde{I}_4$$

(55)

In this manner the results obtained for collineations may be applied directly to polarities. The induced polarity of lines corresponding to $k$ and $K$ is respectively

$$\tilde{I}_6 k = k \tilde{I}_6$$

$$\tilde{I}_6 K = K \tilde{I}_6$$

(56)

(57)

Analogous to (42), induced polarities have the tetrahedron property. For example, using (56) and (42)

$$\tilde{(I_6 k)^T} \Delta (\tilde{I}_6 k) = \tilde{k}^T (\tilde{I}_6 \Delta \tilde{I}_6) \tilde{k}$$

$$= \tilde{k}^T \Delta \tilde{k}$$

$$= \tilde{\mu} \Delta$$

(58)

In Chapter 3 it is shown that polarities may be employed in the development of Euclidean and non-Euclidean geometries using Cayley's conception of the Absolute.