THE ELLIPTIC POLARITY
OF SCREWS

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OBJECTIVE: TO FURTHER THE USE OF CLASSICAL GEOMETRY
IN MODERN KINEMATICS WITH EMPHASIS TOWARDS
THE ANALYSIS AND CONTROL OF ROBOT MANIPULATORS.
DUALITY AND HOMOGENEOUS COORDINATES (PLÜCKER)

- **POINT** - 4 COMPONENTS  \{ DUAL ELEMENTS
- **PLANE** - 4 COMPONENTS  \{ AND DUAL COORDINATES
- **LINE** - 6 COMPONENTS  \{ SELF-DUAL ELEMENT
  \{ AND DUAL COORDINATES

\[ p, \mathbf{P} \]

\[ \mathbf{U} \]

\[ \mathbf{V} \]

- **RAY** LINE COORDINATES - JOIN OF 2 POINTS
  \[ p = |xy| \quad \text{(SIX 2x2 DETERMINANTS)} \]
  \[ \text{COORDINATES OF A FORCE} \]

- **AXIS** LINE COORDINATES - MEET OF 2 PLANES
  \[ p = |uv| \quad \text{(SIX 2x2 DETERMINANTS)} \]
  \[ \text{COORDINATES OF A ROTATION AXIS} \]

- **RAY** <=> **AXIS** - SWITCH FIRST AND LAST 3 COMPONENTS
## Linear Transformations

<table>
<thead>
<tr>
<th>Collineation</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>POINT --- POINT</td>
<td>POINT --- PLANE</td>
</tr>
<tr>
<td>PLANE --- PLANE</td>
<td>PLANE --- POINT</td>
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<tr>
<td>LINE --- LINE</td>
<td>LINE --- LINE</td>
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<td></td>
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<tr>
<td>FORCE --- FORCE</td>
<td>FORCE --- ROTATION</td>
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</tbody>
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Ray and axis coordinates facilitate a distinction between collineations and correlations of lines (self-dual elements).
ABSOLUTE OR INVARIANT POLARITY (CAYLEY)

0 POLARITY - ESTABLISHES AN INVARIANT CONNECTION
OF SPACE BETWEEN DUAL ELEMENTS
(SYMMETRICAL CORRELATION)

0 ELLIPTIC POLARITY - \( \tilde{I}_6 \) (LINES)

\[ p' = \tilde{I}_6 p \quad p' = \tilde{I}_6 p \]

0 \( p' \) AND \( p \) (\( p' \) AND \( p \)) ARE ELLIPTIC POLARS

0 RAY COORDINATES ARE INTERPRETED AS AXIS
COORDINATES AND VICE VERSA

0 ELLIPTIC CONJUGATES (ORTHOGONALITY)

\[ p^t q = 0 \quad p^t Q = 0 \]

0 EUCLIDEAN POLARITY - SINGULAR TRANSFORMATION

0 CLIFFORD'S POLAR OPERATOR \( \omega \) FOR BIQUATERNIONS \( \omega^2 = 1, 0, -1 \)
MECHANICS

0 TWIST ON A SCREW - THE MOST GENERAL INSTANTANEOUS MOTION OF A RIGID BODY

\[
\begin{bmatrix}
\mathbf{v}_0 \\
\mathbf{\Omega}
\end{bmatrix}
\]

0 EXPRESSED IN AXIS COORDINATES

0 WRENCH ON A SCREW - EQUIVALENT TO A SYSTEM OF FORCES AND MOMENTS

\[
\begin{bmatrix}
\mathbf{f} \\
\mathbf{m}_0
\end{bmatrix}
\]

0 EXPRESSED IN RAY COORDINATES

0 RECIPROCAL TWIST AND WRENCH - NO VIRTUAL WORK
SCREWS - GEOMETRICAL ELEMENTS

- **SCREW** - A LINE WITH AN ASSOCIATED PITCH (SCALAR)

- A SELF-DUAL ELEMENT

- A LINEAR COMBINATION OF LINES IS A SCREW

  \[ p = a_1 + \cdots + c, \quad P = aA + \cdots + cC \]

\[
\begin{align*}
\text{WRENCH} & \quad \text{RAY COORDINATES} \\
\text{TWIST} & \quad \text{AXIS COORDINATES}
\end{align*}
\]

- **RECIPROCAL SCREWS** (MIXED COORDINATES)

  \[ p^t q = 0 \quad P^t q = 0 \]

- **ORTHOGONAL SCREWS** (SAME COORDINATES)

  \[ p^t q = 0 \quad P^t q = 0 \]
CONstrained motion

\[ \begin{align*}
\text{AXIS} & \quad \begin{cases}
\text{TWISTS OF FREEDOM:} & \begin{bmatrix}
0 & 0 \\
0 & \Omega_z \\
0 & 0
\end{bmatrix} \\
\text{TWISTS OF NONFREEDOM:} & \begin{bmatrix}
\nu_x & \nu_y & 0 & 0 \\
0 & 0 & \Omega_x & \Omega_y
\end{bmatrix}
\end{cases} \\
\text{RAY} & \quad \begin{cases}
\text{WRENCHES OF CONSTRAINT:} & \begin{bmatrix}
f_x & f_y & 0 & 0 \\
0 & 0 & m_x & m_y
\end{bmatrix} \\
\text{WRENCHES OF NONCONSTRAINT:} & \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\end{cases}
\end{align*} \]

Orthogonal complements

Hybrid manipulator control (Mason, Craig, Raibert)

Simultaneous control of end-effector twists of freedom and wrenches of constraint.
ELLIPITC POLAR SCREWS

0 QUATERNION MAPPING

\[ q = h + r \quad q' = h' + r' \]

\[ p' = q \, p \quad p = q' \, p' \]

0 THE ELLIPITC POLARITY INDUCES THE TRANSFORMATION OF \( q \) INTO ITS INVERSE \( q^{-1} \)

\[ q^{-1} = q' \]
THE QUADRUPLE \((h, r)\)

- **PENCIL OF SCREWS**
  \[(h, r) \quad r \neq 0\]

- **BUNDLE OF SCREWS** (SINGULARITY)
  \[(h, 0) \quad \text{AT THE ORIGIN}\]

- **BUNDLE OF SCREWS** (SINGULARITY)
  \[(\infty, \infty) \quad \text{AT INFINITY}\]
INVERSIVE 4-SPACE

0 POINT ( h, r )

0 ELLIPTIC POLAR
  POINT ( h', r' )

0 \(( h^2 + r^2 )( h'^2 + r'^2 ) = 1\)

0 THE ELLIPTIC POLARITY INDUCES
  AN INVERSION THROUGH THE
  HYPERSPHERE \(( h^2 + r^2 ) = 1\)
  FOLLOWED BY A REFLECTION IN
  HYPERPLANE \( h = 0 \)

INVERSION THROUGH THE UNIT SPHERE \(( h = 0 \) )
HELIX MAPPING

\[
\begin{align*}
q &= h + r = \tau' - \kappa' \\
q' &= h' + r' = \tau - \kappa
\end{align*}
\]

\[\tau \text{ - TORSION}\]

\[\kappa \text{ - CURVATURE VECTOR THROUGH ORIGIN}\]
TWO SYSTEM PLANAR REPRESENTATION

0 BALL'S \((h, r)\) PLANE

0 POINT - SCREW

0 CIRCLE - CYLINDROID

0 EXTENSION TO \((h, r)\) COMPLEX PLANE

0 POINT - PENCIL OF SCREWS

\[ q = h + ir \] (SPECIAL QUATERNION)

0 CIRCLE - PENCIL OF CYLINDROIDS

\[ \| q - \gamma \| = r^2, \quad \gamma = \alpha + i\beta \]

0 ELLIPTIC POLARITY

\[ q' = q^{-1} \]
SELF-POLAR CYLINDROIDS

$$0 \quad \| q - \gamma \| = r^2 \quad \text{CYLINDROID MAPPING}$$

$$0 \quad \| q^{-1} - \gamma \| = r^2 \quad \text{POLAR CYLINDROID MAPPING}$$

EVERY CYLINDROID OF THE PENCIL IS SELF-POLAR

ONLY PENCIL OF CYLINDROIDS IS SELF-POLAR
EFFECT OF ORIGIN TRANSLATION ON THE POLAR SCREW

FIG. 10